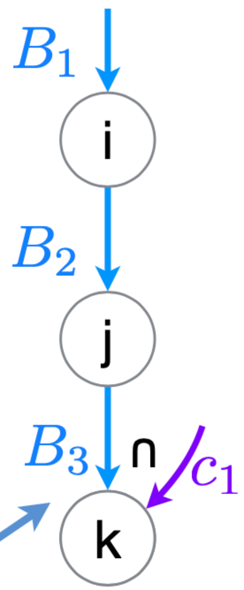
Notation

Pat Hanrahan

"Semantics, Not Syntax"

$$A_{ij} = \sum_k B_{ijk} c_k$$




```
for (int i = 0; i < m; i++) {

  for (int pB2 = B2_pos[i]; pB2 < B2_pos[i+1]; pB2++) {
    int j = B2_crd[pB2];


    int pA2 = i*n + j;
    int pB3 = B3_pos[pB2];
    int pc1 = c1_pos[0];
    while (pB3 < B3_pos[pB2+1] && pc1 < c1_pos[1]) {
      int kB = B3_crd[pB3];
      int kc = c1_crd[pc1];
      int k = min(kB, kc);
      if (kB == k && kc == k) {
        A[pA2] += B[pB3] * c[pc1];
      }
      if (kB == k) pB3++;
      if (kc == k) pc1++;

    }
  }
}
```

Key operation is to coiterate over data structures

Intersection coiteration

Concrete index notation specifies order of computations and location of intermediate values

$$A_{ij} = B_{ij} + C_{ij} \longrightarrow \forall_i \forall_j \; A_{ij} = B_{ij} + C_{ij}$$

$$\alpha = \sum_i b_i c_i \longrightarrow \forall_i \; \alpha \mathrel{+}= b_i c_i$$

$$a_i = \sum_j B_{ij} c_j \longrightarrow \forall_i \forall_j \; a_i = t \textbf{ where } t \mathrel{+}= B_{ij} c_j$$

THE FOUNDATION OF THE GENERAL THEORY OF RELATIVITY

By A. EINSTEIN

"*Contraction*" *of a Mixed Tensor.*—From any mixed tensor we may form a tensor whose rank is less by two, by equating an index of covariant with one of contravariant character, and summing with respect to this index ("contraction"). Thus, for example, from the mixed tensor of the fourth rank $A^{\sigma\tau}_{\mu\nu}$, we obtain the mixed tensor of the second rank,

$$A^{\tau}_{\nu} = A^{\mu\tau}_{\mu\nu} \quad \left(= \sum_{\mu} A^{\mu\tau}_{\mu\nu} \right),$$

and from this, by a second contraction, the tensor of zero rank,

$$A = A^{\nu}_{\nu} = A^{\mu\nu}_{\mu\nu}.$$

The theory which is presented in the following pages conceivably constitutes the
farthest-reaching generalization of a theory which, today, is generally called the
[1] "theory of relativity"; I will call the latter one—in order to distinguish it from the
[2] first named—the "special theory of relativity," which I assume to be known. The
generalization of the theory of relativity has been facilitated considerably by
Minkowski, a mathematician who was the first one to recognize the formal
[3] equivalence of space coordinates and the time coordinate, and utilized this in the
construction of the theory. The mathematical tools that are necessary for general
relativity were readily available in the "absolute differential calculus," which is based
upon the research on non-Euclidean manifolds by Gauss, Riemann, and Christoffel,
and which has been systematized by Ricci and Levi-Civita and has already been
[4] applied to problems of theoretical physics. In section B of the present paper I
developed all the necessary mathematical tools—which cannot be assumed to be
known to every physicist—and I tried to do it in as simple and transparent a manner
as possible, so that a special study of the mathematical literature is not required for
[5] the understanding of the present paper. Finally, I want to acknowledge gratefully my
friend, the mathematician Grossmann, whose help not only saved me the effort of
[6] studying the pertinent mathematical literature, but who also helped me in my search
for the field equations of gravitation.

Representations

Herb Simon

Nobel Prize in Economics (1977)
“for his pioneering research into the decision-making process within economic organizations”

Turing Award (1975)
"basic contributions to artificial intelligence, the psychology of human cognition, and list processing"

Number Scrabble

Goal: Pick three numbers that sum to 15

1 2 3 4 5 6 7 8 9

A:

B:

Number Scrabble

Goal: Pick three numbers that sum to 15

1 6 7 9

A: 8 4 5

B: 2 3 ?

Number Scrabble

Goal: Pick three numbers that sum to 15

1 2 3 4 5 6 7 8 9

A: 8 4 5

B: 2 3 ?

Number Scrabble

Goal: Pick three numbers that sum to 15

1 2 3 4 5 6 7 9

A: 8

B:

Number Scrabble

Goal: Pick three numbers that sum to 15

1 3 4 5 6 7 9

A: 8

B: 2

Number Scrabble

Goal: Pick three numbers that sum to 15

1 3 5 6 7 9

A: 8 4

B: 2

Number Scrabble

Goal: Pick three numbers that sum to 15

1 5 6 7 9

A: 8 4

B: 2 3

Number Scrabble

Goal: Pick three numbers that sum to 15

1 6 7 9

A: 8 4 5

B: 2 3 ?

Problem Isomorphs

Problem Isomorph

4	3	8
9	5	1
2	7	6

Magic Square: All rows, columns, diagonals sum to 15

Transform to Tic-Tac-Toe

4	3	8
9	5	1
2	7	6

Transform to Tic-Tac-Toe

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Transform to Tic-Tac-Toe

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“Why is a Picture (Sometimes) Worth 10,000 Words”

Larkin and Simon, Cognitive Science, 1987

Why?

Reduce memory load

- Working memory is limited
- Store information in the diagram

Reduce search time

- Pre-attentive (constant-time) search process
- Spatially-indexed patterns store the "facts"

Allow perceptual inference

- Map inference to pattern finding

The Representation Effect

Although two representations
may be equivalent,
one is often much "better"
for a given problem

"Better" means

- Faster
- Fewer errors
- Better comprehension
- ...

Summary of Financial Performance

		Central		East		South		West	
		Total Sales	Total Profit	Total Sales	Total Profit	Total Sales	Total Profit	Total Sales	Total Profit
Coffee	Amaretto	$14,011	5,105	$2,993	1,009			$9,265	-1,225
	Columbian	$28,913	8,528	$47,386	27,253	$21,664	8,767	$30,357	11,253
	Decaf Irish Cream	$26,155	9,632	$6,261	2,727	$11,592	2,933	$18,235	-1,305
Espresso	Caffe Latte					$15,442	3,872	$20,458	7,502
	Caffe Mocha	$35,218	14,640	$16,646	-6,230	$14,163	5,201	$18,876	4,064
	DecafEspresso	$24,485	8,860	$7,722	2,410	$15,384	5,930	$30,578	12,302
	RegularEspresso			$24,036	10,062				
Herbal Tea	Chamomile	$36,570	14,434	$2,194	765	$11,186	3,180	$25,632	8,852
	Lemon	$21,978	6,251	$27,176	7,901	$14,497	2,593	$32,274	13,120
	Mint	$9,337	4,069	$11,992	-2,242			$14,380	4,330
Tea	Darjeeling	$30,289	10,772	$14,096	6,497			$28,769	11,780
	Earl Grey	$32,881	10,331	$6,505	3,405			$27,387	10,425
	Green Tea	$5,211	1,227	$11,571	5,654			$16,063	-7,109

How much mint tea was sold in the west?

Summary of Financial Performance									
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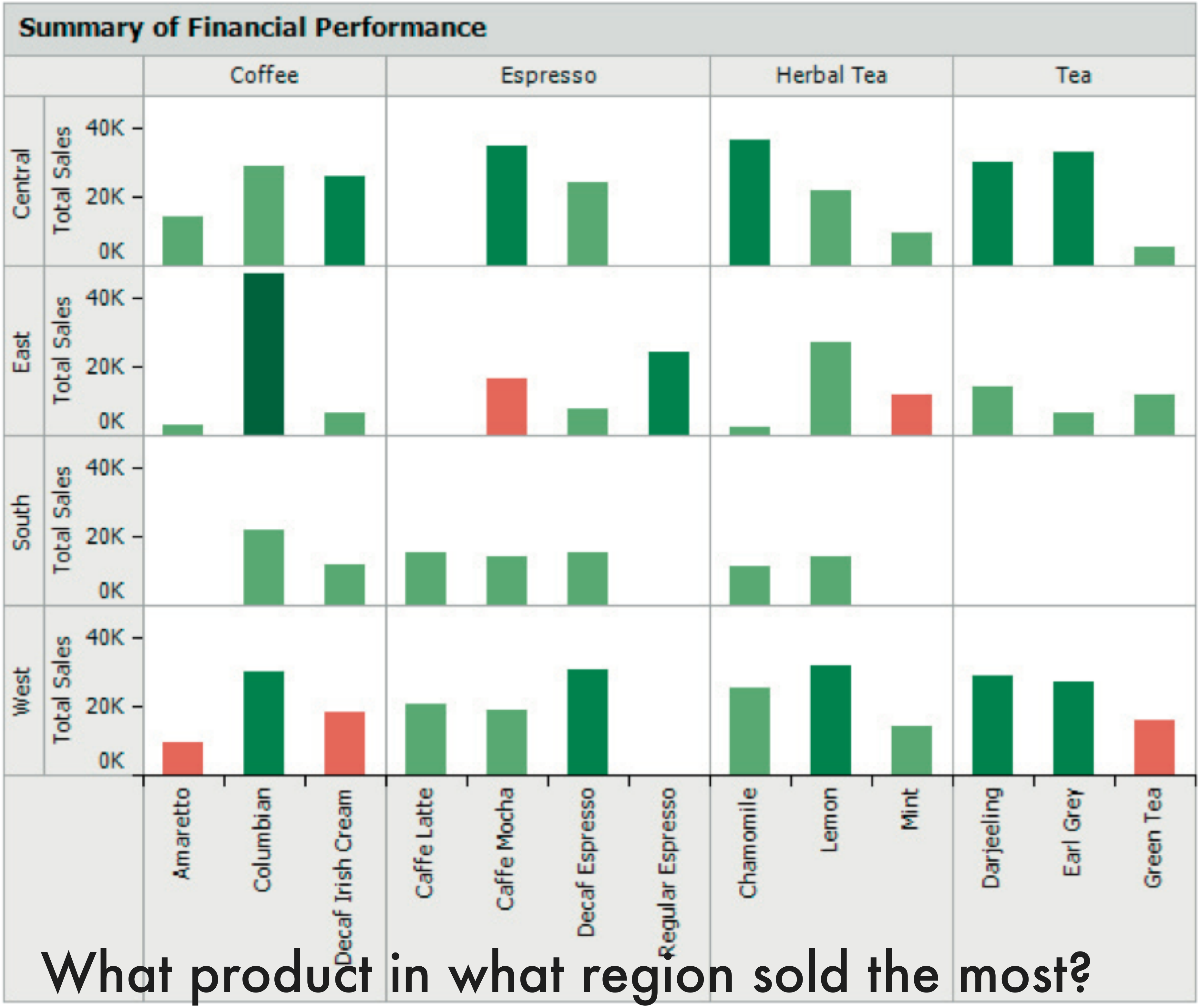
What product in what region sold the most?



What product in what region sold the most?

“Number Representations”

Zhang and Norman

Number Representations

Counting – Tallying

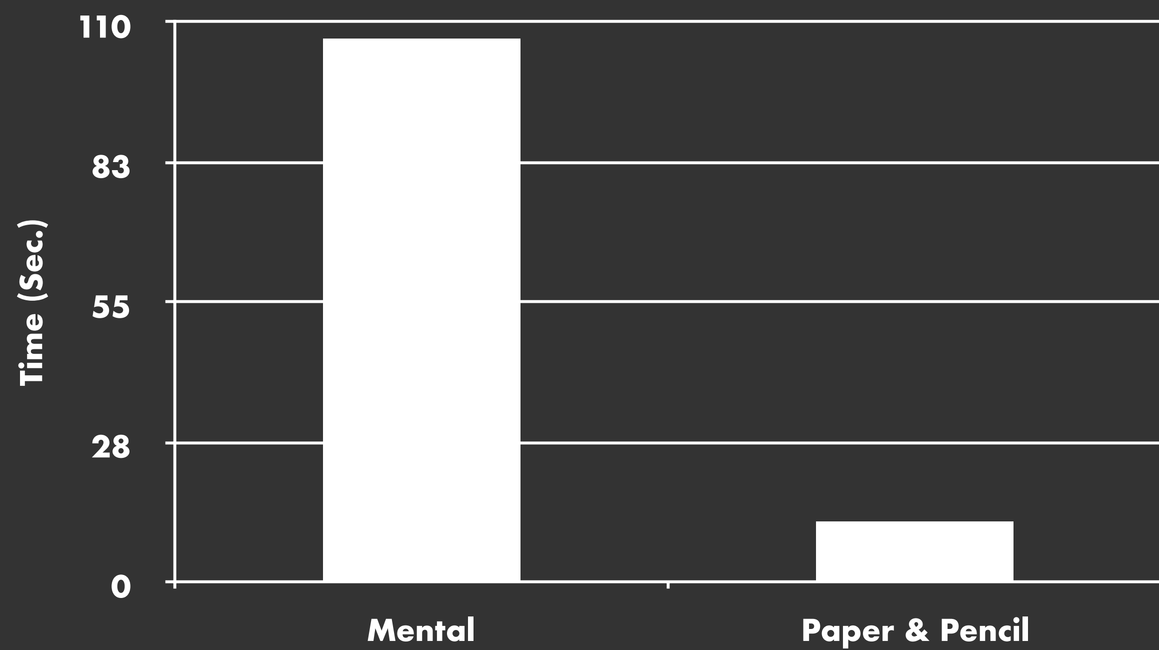
Adding – Roman numerals

XXIII + XII = XXXIIIII = XXXV

Multiplication – Arabic number systems

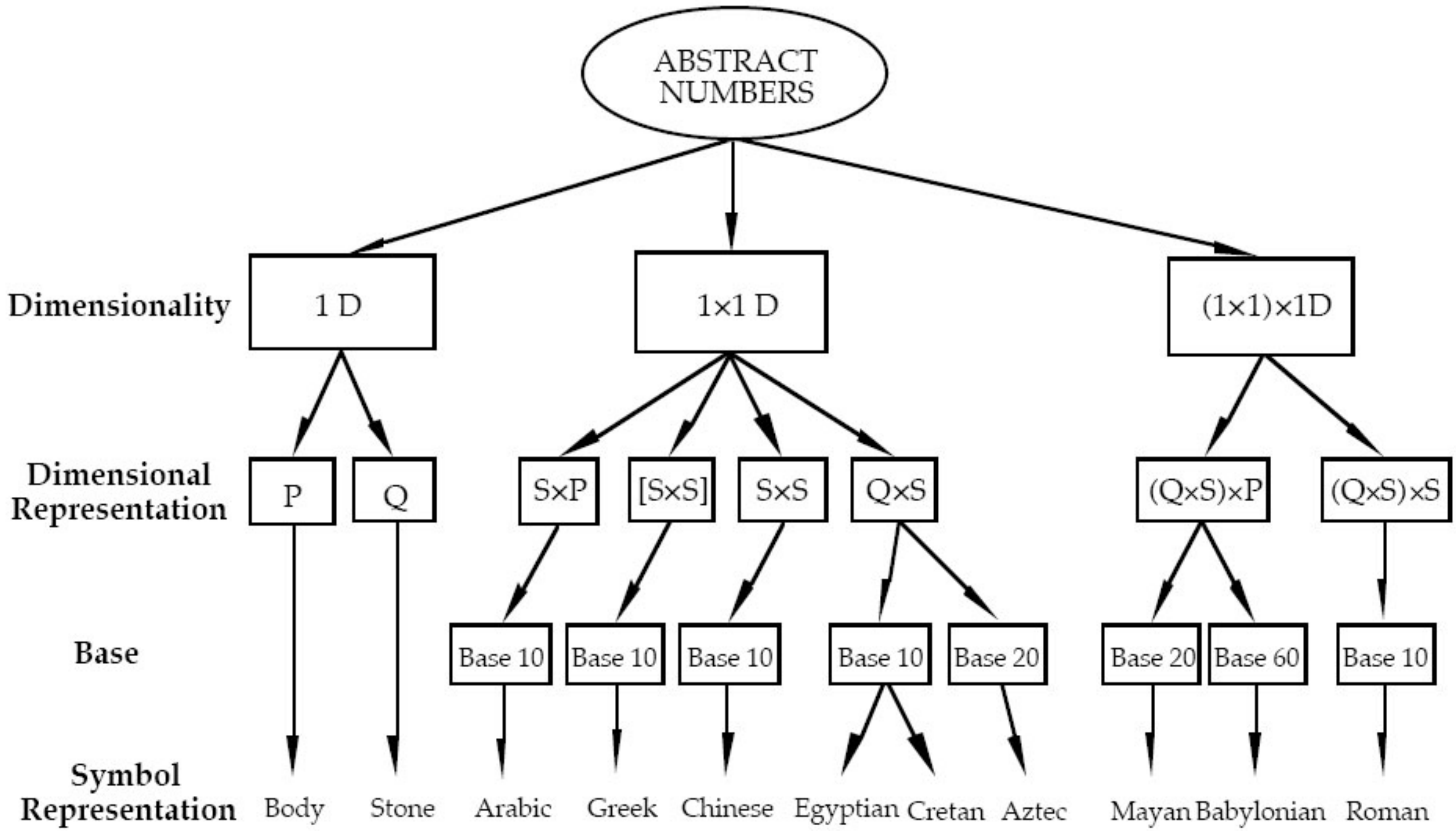
Long-Hand Multiplication

34
x 72
68
238
2448



From "Introduction to Information Visualization,"

Card, Schneiderman, Mackinlay



Zhang and Norman, The Representations of Numbers,

Cognition, 57, 271-295, 1996

Distributed Cognition

External (E) vs. Internal (I) process

		Roman	Arabic
1.	**Separate power & base**	**I**	**E**
2.	**Get base value**	**E**	**I**
3.	**Multiply base values**	**I**	**I**
4.	**Get power values**	**I**	**E**
5.	**Add power values**	**I**	**E**
6.	**Combine base & power**	**I**	**E**
7.	**Add results**	**I**	**E**

Arabic more efficient than Roman

Set Representations

Representation	Union	Find
Cons	$O(n^2)$	$O(n)$
Sorted Cons	$O(n)$	$O(\log n)$
Hash	*$O(n^2)$*	*$O(1)$*
Bit Vector	$O(n)$	$O(1)$
Union-Find Forest	$O(\alpha(n))$	$O(\alpha(n))$

An Automatic Technique for Selection of Data Representations in SETL Programs, Schonberg, Schwarz, Sharir, 1981

Notation as a Tool of Thought

Kenneth Iverson

Notation as a Tool for Thought

"The thesis of the present paper is that the advantages of executabiity and universality found in programming languages can be effectively combined, in a single coherent language, with the advantages offered by mathematical notation"

K. Iverson

Arithmetic and Algebra in APL (k)

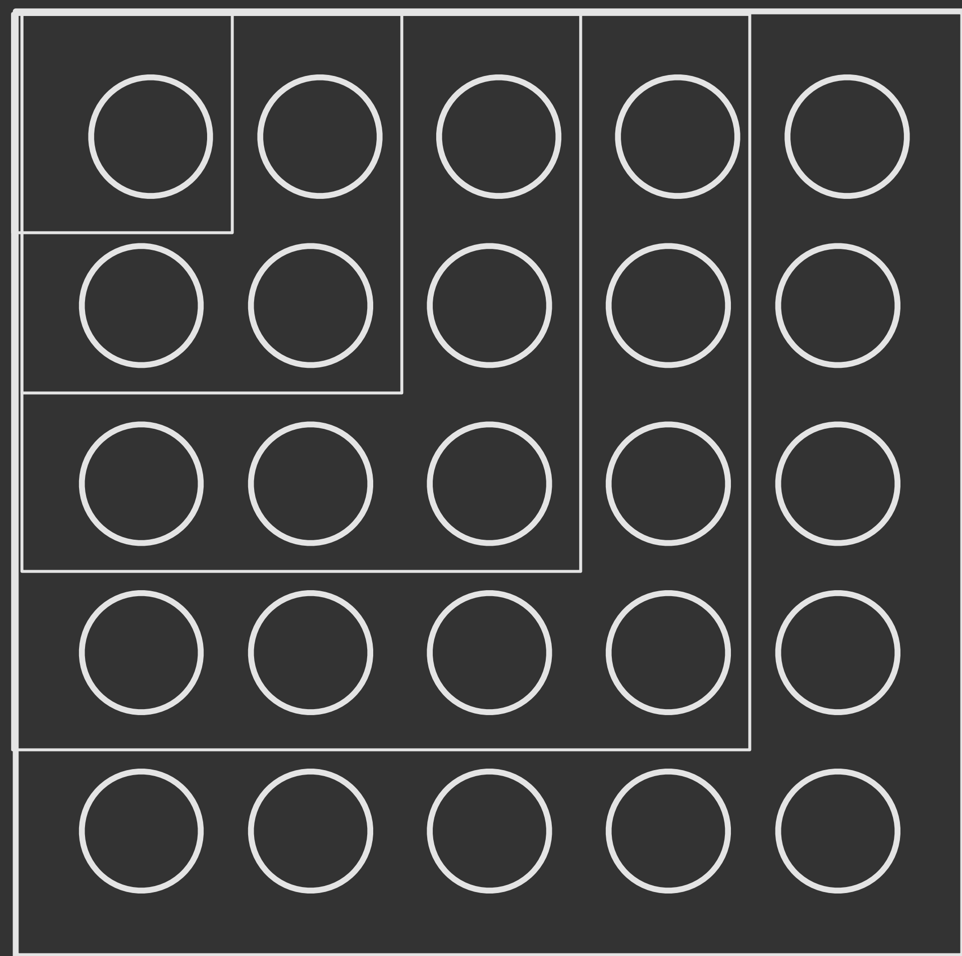
```
> k = 5
> til k
   0 1 2 3 4
> 1 + 2 * til k
   1 3 5 7 9
> +/ 1+2*til k          // 1 + 3 + 5 + 7 + 9
   25
> k*k
    25
```

Program Transformations as Proofs

```
// sum of k odd numbers
   +/ (1 + 2 * til k)
== // definition of multiplication
   +/ (1 + (til k) + (til k))
== // addition is commutative and associative
   +/ (1 + (til k) + (reverse til k))
== // 0 1 2 + 2 1 0 = (0+2) + (1+1) + (2+0) = 2+2+2 // +/ k # k-1
   +/ (1 + k # (k-1))
== // scalar + vector causes scalar to be repeated k times
   +/ k#k
== // definition of multiplication as repeated addition: k*k = +/ k#k
   // e.g. 3*3 = +/ 3 3 3      ,
   k*k
```

K. Iverson, *Arithmetic*, 1991

Visual Proofs



Algebra

$1+3+5+7+9=5^2$

Characteristics of Notation

Executable mathematics

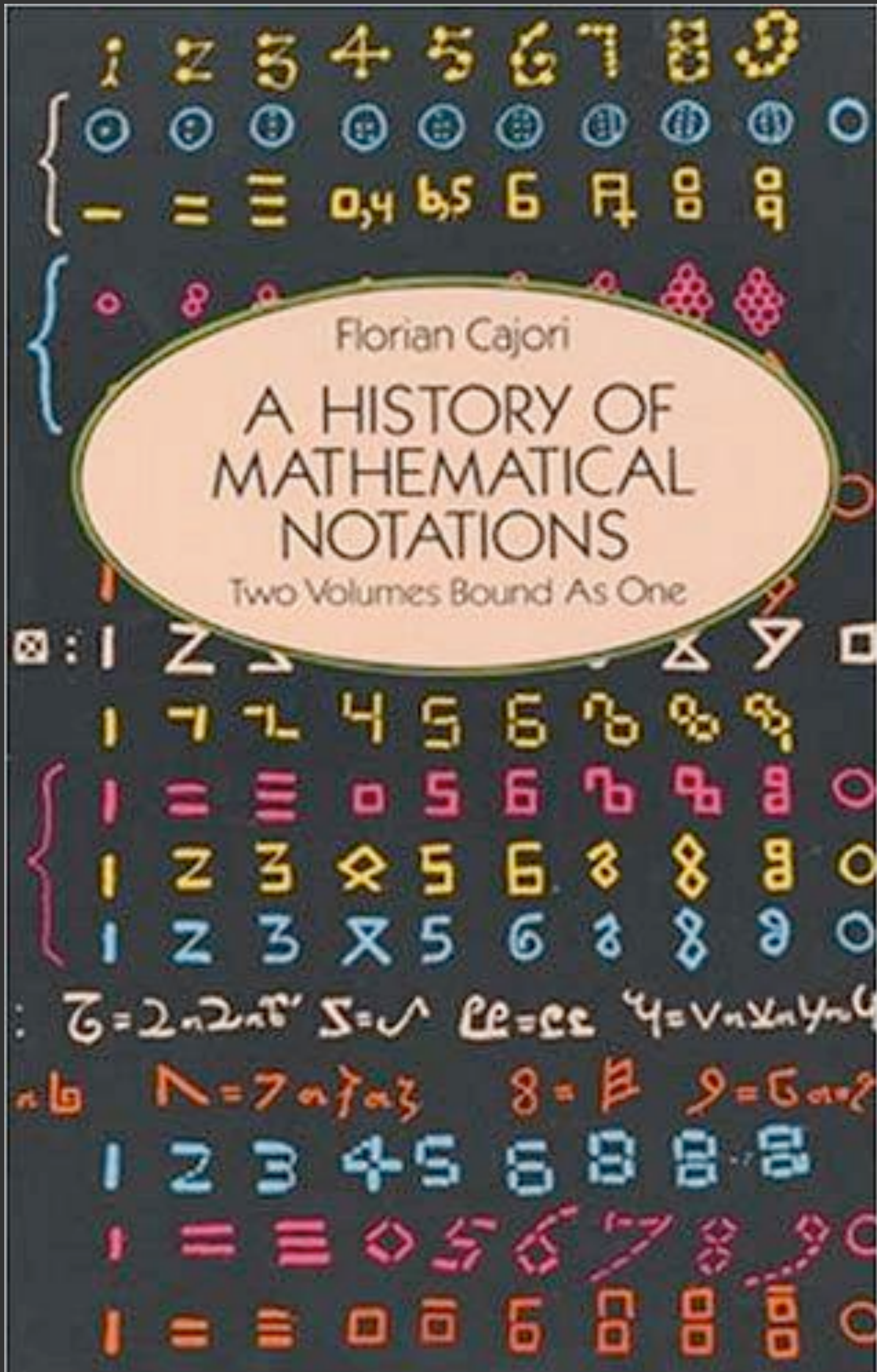
1 Ease of expression

2 Suggestivity (naturalness)

3 Subordinate detail (abstract)

4 Economy (concise)

5 Formal (proofs)



Florian Cajori (1859-1930)

History of Mathematics

University of California,

Berkeley

Mathematica

Rich character and symbol set (2500)

Multiple versions of a symbol to resolve ambiquity

- *i*
- double struck i = sqrt(-1)

Two-dimensional (over, subscript, ...)

StandardForm, TraditionalForm, ...

- 100 heuristic to go from traditional to standard

Input macros (map ascii to symbol, ->)

Mathematical Notation: Past and Future, Stephen Wolfram, 2000

The Incredible Convenience of Mathematica Image Processing

Theodore Gray

Supporting Notation

```
// Minimal syntax - s expressions

// Lisp
(cond
  ((= n 10) (= m 1))
  ((> n 10) (= m 2) (= n (* n m)))
  ((< n 10) (= n 0)))

// Symbols are characters delimited by
// spaces and punctuation

// Special forms

// Minimal notation!
```

```
// Smalltalk, Ruby
employee name first

// unary operators / methods
// parsed left to right
// left associative
//   = employee.name.first

// binary operators
// left associative, no precedence
1 + 2 * 3 = (1 + 2) * 3

// unary operators have precedence over
binary
1 + 4 sqrt = 1 + (4 sqrt)
```

```
// APL (k)
+/ 1 + 2 * til k

// monadic (til) and dyadic (+ *)
//
// right associative
// functions have same precedence
// operators (higher-order functions)
// operators > functions
+/ (1 + (2 * (til k)))
```

```
// Haskell

// sections
(+) 1 2 = 1 + 2

// currying, left associative
(((+) 1) 2)

// precedence and associativity
infixl 6 +
```

Unicode Symbols

- From the `Prelude.Unicode` module
  - Values
    - `not` = `(¬)`
    - `(&&)` = `(∧)`
    - `(||)` = `(∨)`
    - `(==)` = `(≡)`
    - `(/=)` = `(≠)` = `(≢)`
    - `<=` = `≤` = `≯`
    - `>=` = `≥` = `≮`
    - `pi` = `π`
    - `(/)` = `(÷)`
    - `(*)` = `(⋅)`
    - `(.)` = `(∘)`
    - `(++)` = `(⧺)`
    - `elem` = `(∈)`
    - `notElem` = `(∉)`
    - `undefined` = `(⊥)`
  - Types
    - `Integer` = `ℤ`
    - `Rational` = `ℚ`

```
// Agda

// define operators with blanks for arguments
_+_ x y =
_+_ x y = x + y

// precedence and associativity
infixl 6 _+_

// statement forms
if_then_else_ x y z =
if x then y else

// unicode characters in names
// identifiers must be separated with spaces
```

Category