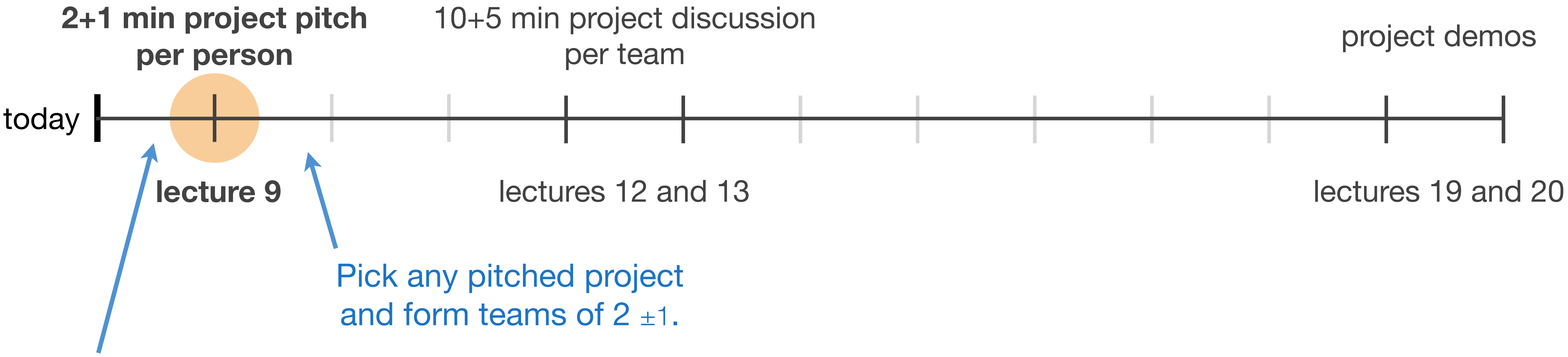
Lecture 8 - Sparse Iteration Theory II

Stanford CS343D (Fall 2021)
Fred Kjolstad

Course Project



2+1 min project pitch per person

10+5 min project discussion per team

project demos

today

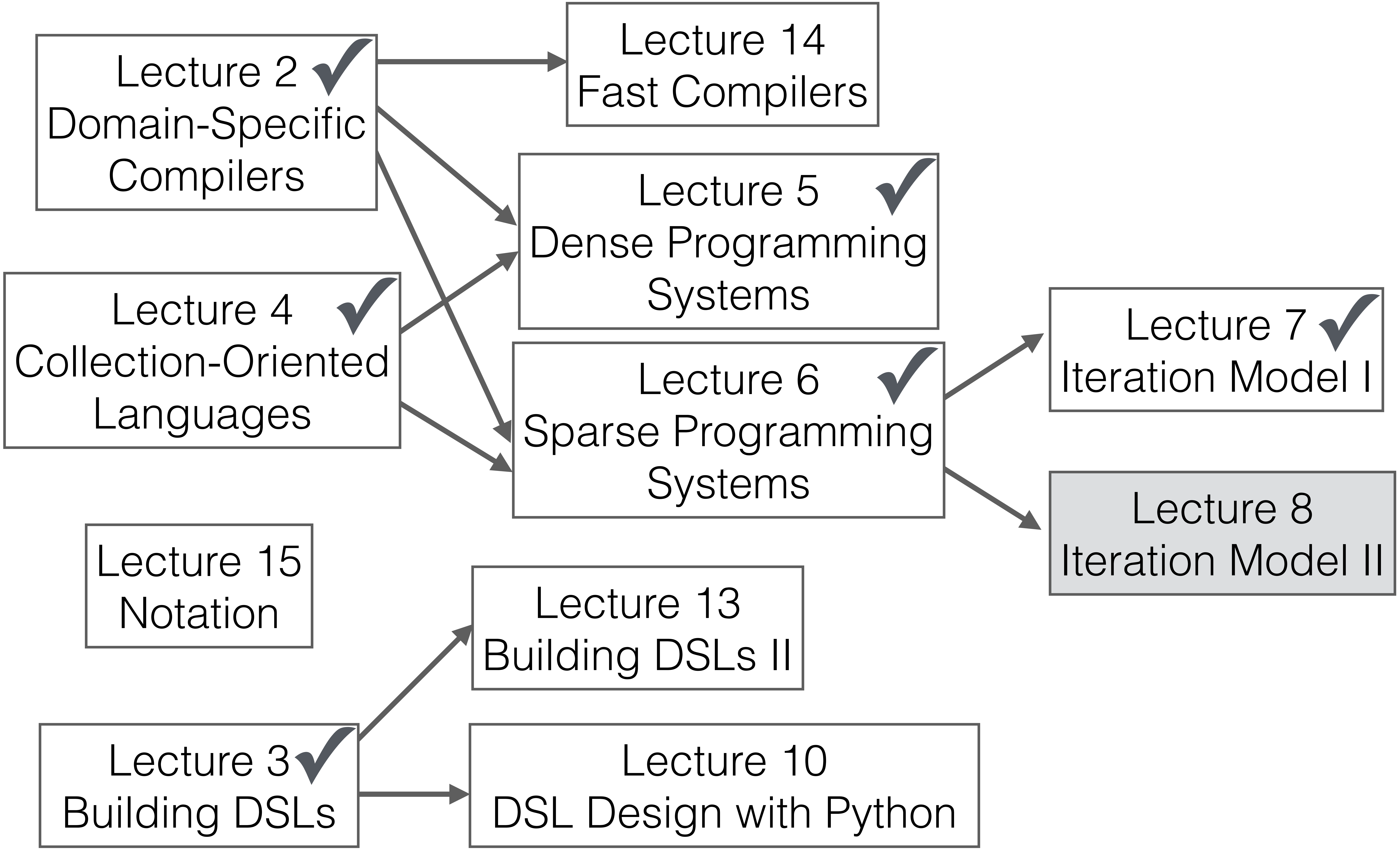
lecture 9

lectures 12 and 13

lectures 19 and 20

Pick any pitched project and form teams of 2 ±1.

Each person contributes one pitch slide to a google slide deck. These pitches are not binding.



Lecture 2 ✓
Domain-Specific Compilers

Lecture 14
Fast Compilers

Lecture 5 ✓
Dense Programming Systems

Lecture 4 ✓
Collection-Oriented Languages

Lecture 6 ✓
Sparse Programming Systems

Lecture 7 ✓
Iteration Model I

Lecture 8
Iteration Model II

Lecture 15
Notation

Lecture 13
Building DSLs II

Lecture 3 ✓
Building DSLs

Lecture 10
DSL Design with Python

Overview of topics

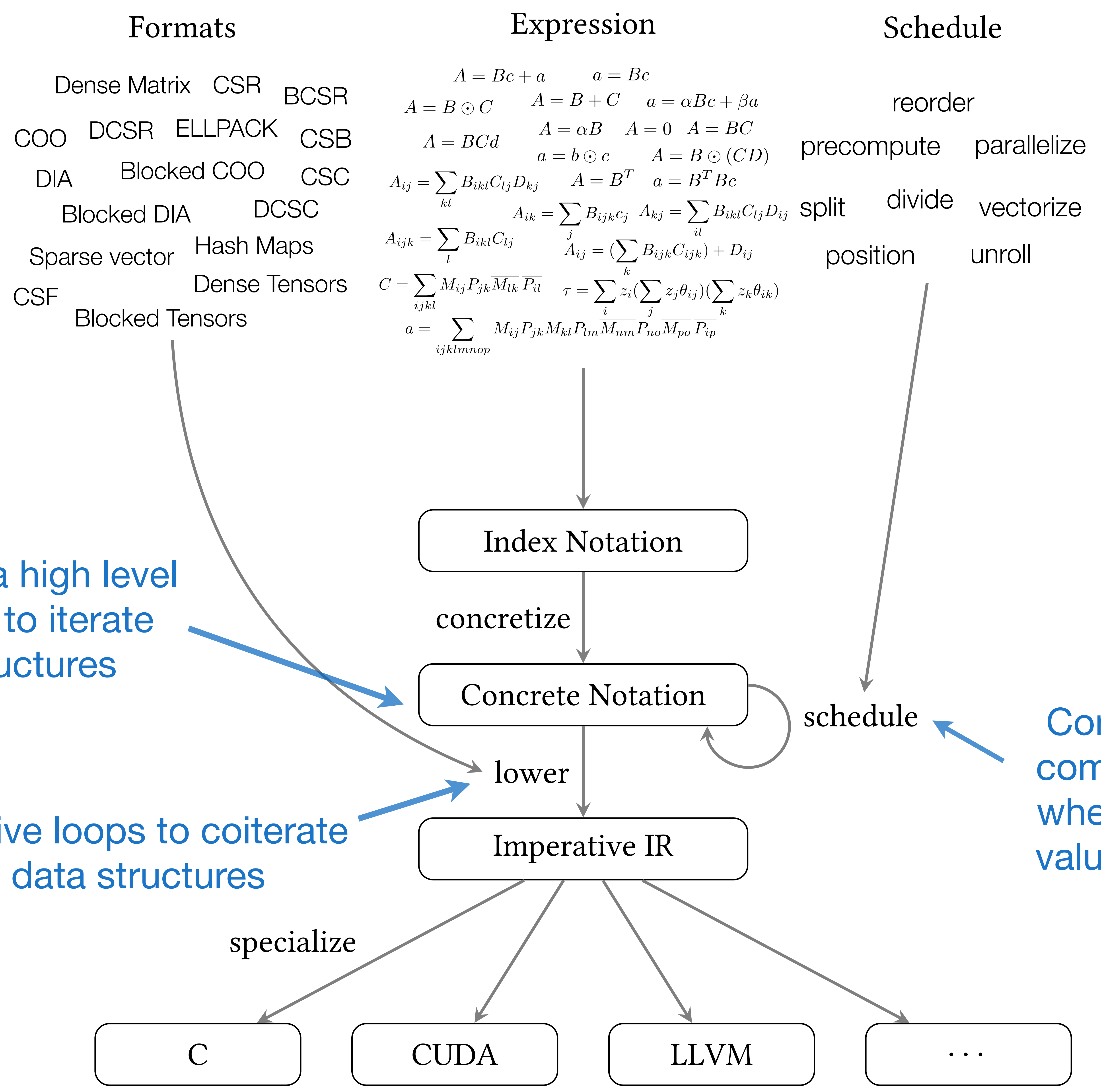
Lecture 7

- Data representation
- Iteration spaces
- Iteration graph IR
- Iteration lattices to represent coiteration

Lecture 8

- Concrete index notation IR
- Code generation algorithm
- Derived iteration spaces
- Optimizing transformations

Overview of compilation stages



Formats

Dense Matrix
CSR
BCSR
COO
DCSR
ELLPACK
CSB
DIA
Blocked COO
CSC
Blocked DIA
DCSC
Sparse vector
Hash Maps
CSF
Dense Tensors
Blocked Tensors

Expression

$A = Bc + a$ $\quad a = Bc$

$A = B \odot C$ $\quad A = B + C$ $\quad a = \alpha Bc + \beta a$

$A = BCd$ $\quad A = \alpha B$ $\quad A = 0$ $\quad A = BC$

$a = b \odot c$ $\quad A = B \odot (CD)$

$A_{ij} = \sum_{kl} B_{ikl} C_{lj} D_{kj}$ $\quad A = B^T$ $\quad a = B^T Bc$

$A_{ik} = \sum_{j} B_{ijk} c_j$ $\quad A_{kj} = \sum_{il} B_{ikl} C_{lj} D_{ij}$

$A_{ijk} = \sum_{l} B_{ikl} C_{lj}$ $\quad A_{ij} = (\sum_{k} B_{ijk} C_{ijk}) + D_{ij}$

$C = \sum_{ijkl} M_{ij} P_{jk} \overline{M_{lk}}\, \overline{P_{il}}$ $\quad \tau = \sum_{i} z_i (\sum_{j} z_j \theta_{ij})(\sum_{k} z_k \theta_{ik})$

$a = \sum_{ijklmnop} M_{ij} P_{jk} M_{kl} P_{lm} \overline{M_{nm}} P_{no} \overline{M_{po}}\, \overline{P_{ip}}$

Schedule

reorder
precompute
parallelize
split
divide
vectorize
position
unroll

Index Notation

concretize

Concrete Notation

schedule

lower

Imperative IR

specialize

C
CUDA
LLVM
. . .

Express computation at a high level before generating code to iterate over specific data structures

Generate imperative loops to coiterate over relevant data structures

Control order of computation and where temporary values are stored

Concrete index notation specifies order of computations and location of intermediate values

$$A_{ij} = B_{ij} + C_{ij} \quad\longrightarrow\quad \forall_i \forall_j \; A_{ij} = B_{ij} + C_{ij}$$

$$\alpha = \sum_i b_i c_i \quad\longrightarrow\quad \forall_i \; \alpha \mathrel{+}= b_i c_i$$

$$a_i = \sum_j B_{ij} c_j \quad\longrightarrow\quad \forall_i \; a_i = t \;\textbf{ where }\; \forall_j \; t \mathrel{+}= B_{ij} c_j$$

Concrete index notation grammars

Assignment statement

$A_{i\dots} = \text{expr}$

Forall statement

$\forall_i \ \text{stmt}$

Where statement

$\text{stmt}_c \ \textbf{where} \ \text{stmt}_p$

Environment

index index “$\xrightarrow{\text{“collapse”}}$” index

index “$\xrightarrow{\text{“split(” d “,” s “)”}}$” index index

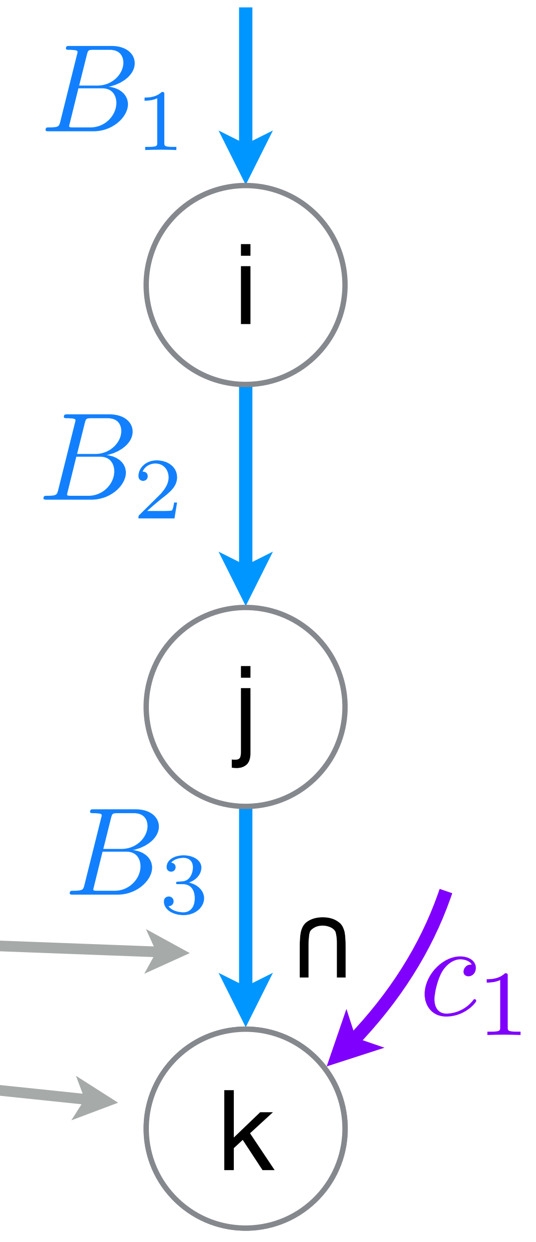
“bound(” index “,” b “)”

“parallelize(” index “,” p “,” r “)”

“unroll(”index “,” u“)”

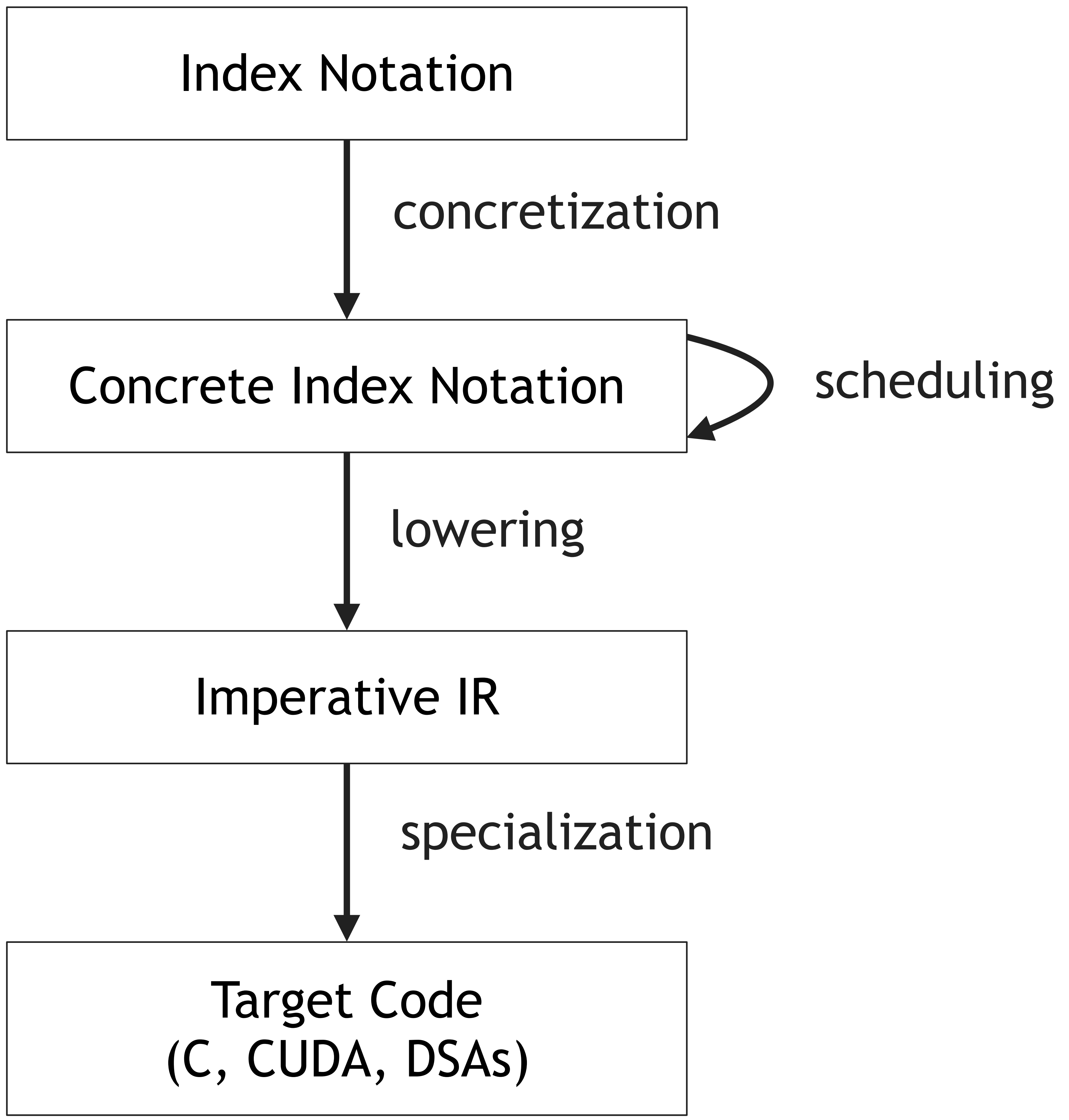
Concrete index notation contains iteration graphs

$$\forall_i \forall_j \forall_k\ A_{ij} \mathrel{+}= B_{ijk}\, c_k$$



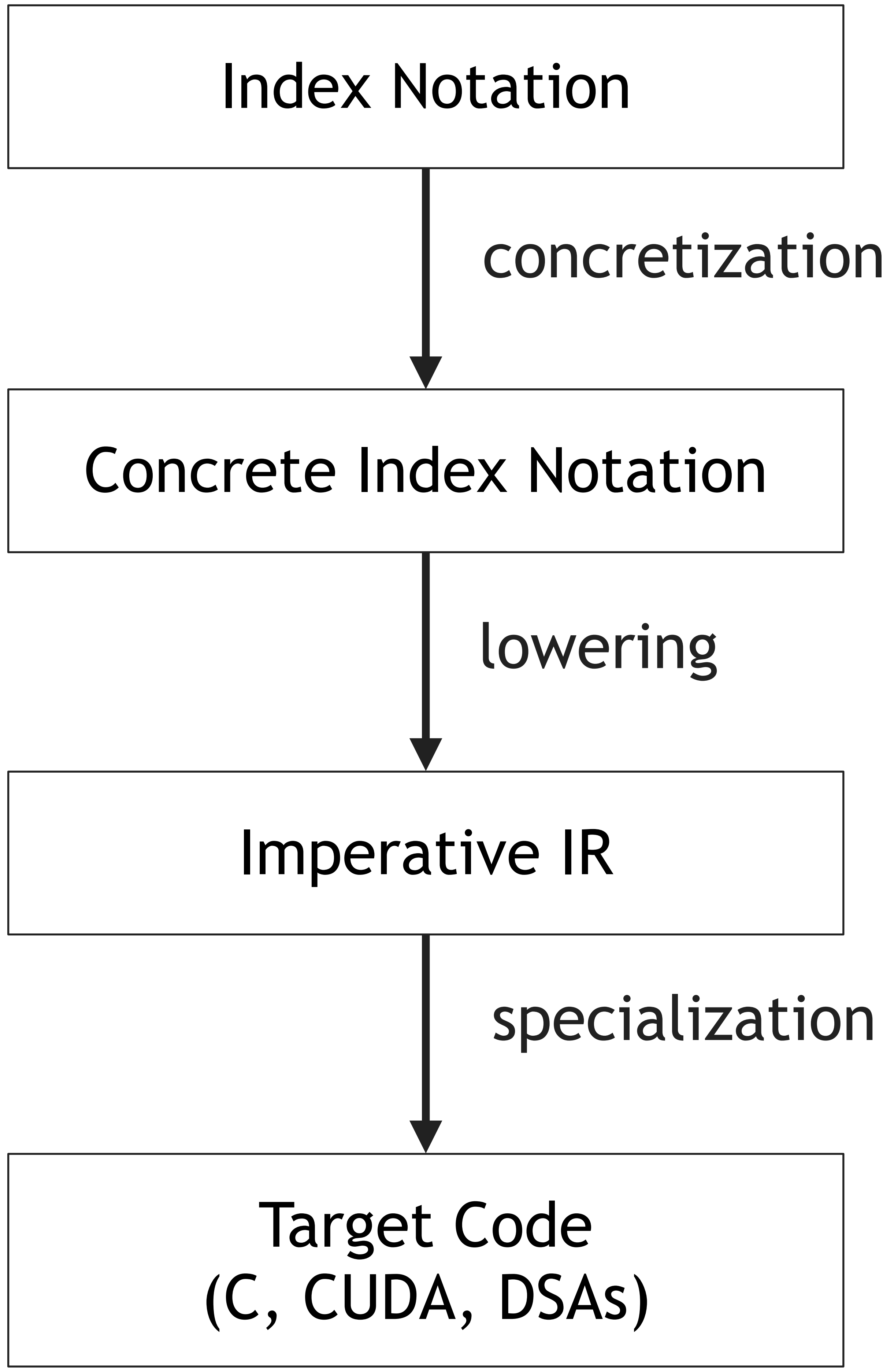
$$B_{ijk} \cap c_k$$

Concrete index notation as an optimization IR



Optimizing transformations are applied before coiteration code is generated. Thus it applies equally to dense and sparse computation

Concrete index notation example



Index Notation

concretization

Concrete Index Notation

lowering

Imperative IR

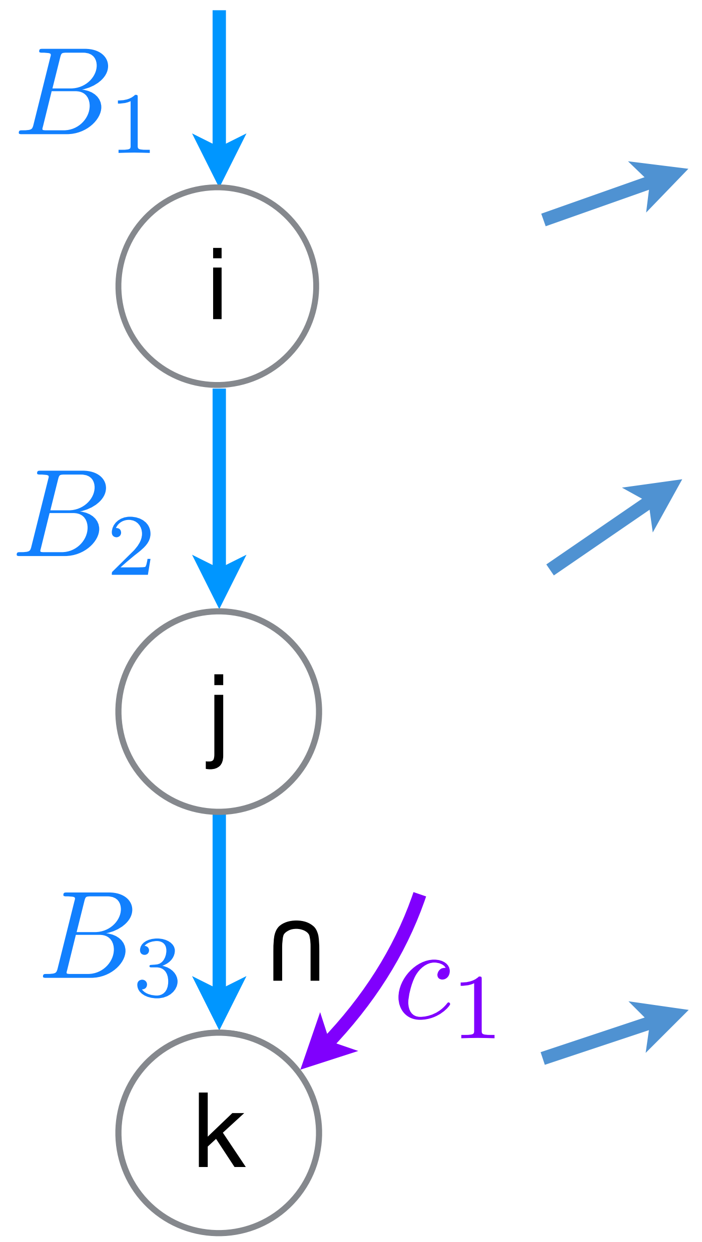
specialization

Target Code
(C, CUDA, DSAs)

$$A_{ij} = \sum_k B_{ijk} c_k$$

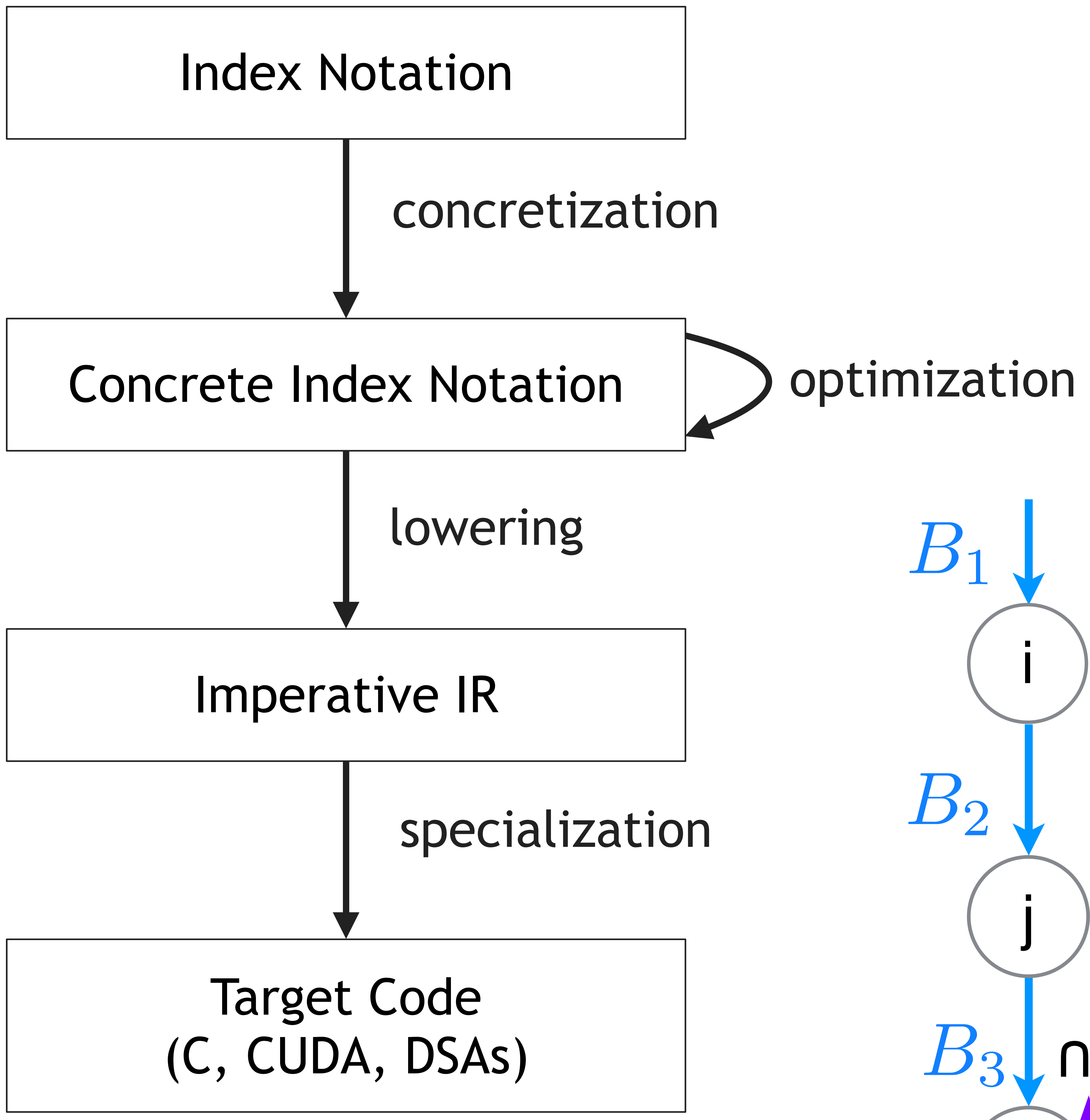
Order of computation

$$\forall_i \forall_j \forall_k \; A_{ij} \mathrel{+}= B_{ijk} c_k$$



```
for (int i = 0; i < m; i++) {

  for (int pB2 = B2_pos[i]; pB2 < B2_pos[i+1]; pB2++) {
    int j = B2_crd[pB2];


    int pA2 = i*n + j;
    int pB3 = B3_pos[pB2];
    int pc1 = c1_pos[0];
    while (pB3 < B3_pos[pB2+1] && pc1 < c1_pos[1]) {
      int kB = B3_crd[pB3];
      int kc = c1_crd[pc1];
      int k = min(kB, kc);
      if (kB == k && kc == k) {
        A[pA2] += B[pB3] * c[pc1];
      }
      if (kB == k) pB3++;
      if (kc == k) pc1++;

    }
  }
}
```

Concrete index notation example

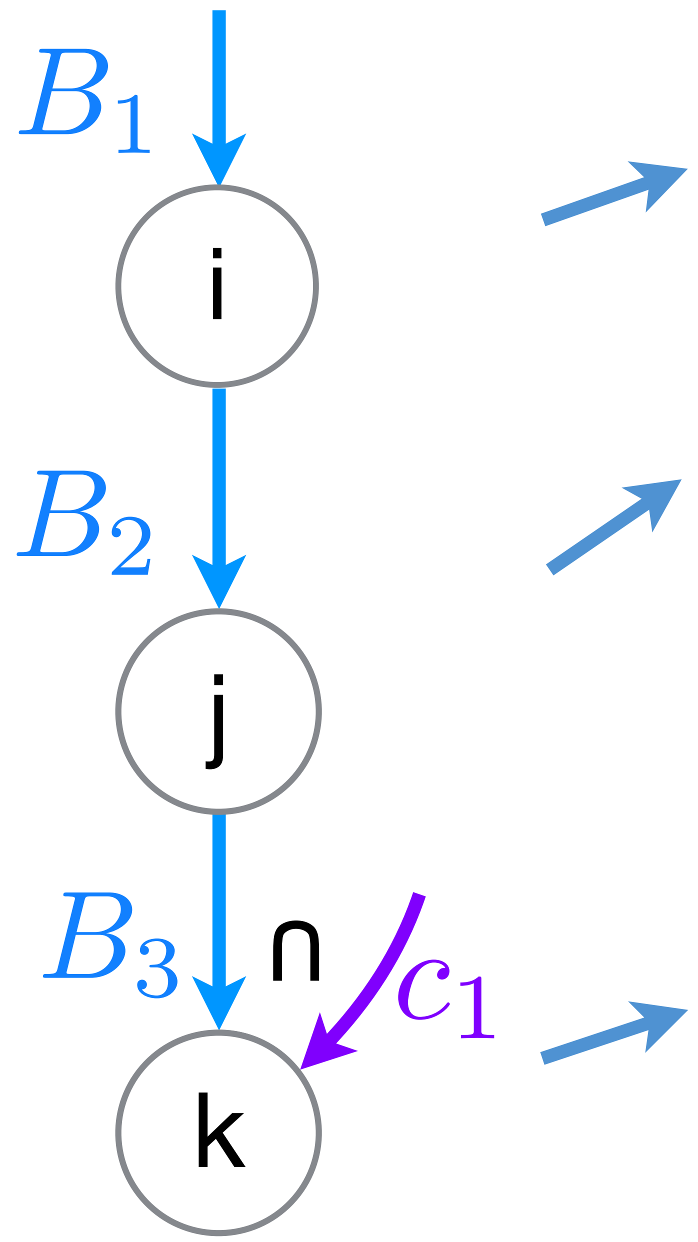


$$A_{ij} = \sum_k B_{ijk} c_k$$

Order of computation

Temporary storage

$$\forall_i \forall_j \left(A_{ij} = t\right) \textbf{ where } \left(\forall_k\, t \mathrel{+}= B_{ijk} c_k\right)$$



```
for (int i = 0; i < m; i++) {

  for (int pB2 = B2_pos[i]; pB2 < B2_pos[i+1]; pB2++) {
    int j = B2_crd[pB2];

    float t = 0.0;
    int pA2 = i*n + j;
    int pB3 = B3_pos[pB2];
    int pc1 = c1_pos[0];
    while (pB3 < B3_pos[pB2+1] && pc1 < c1_pos[1]) {
      int kB = B3_crd[pB3];
      int kc = c1_crd[pc1];
      int k = min(kB, kc);
      if (kB == k && kc == k) {
        t += B[pB3] * c[pc1];
      }
      if (kB == k) pB3++;
      if (kc == k) pc1++;
    A[pA2] = t;
    }
  }
}
```

Workspace to scatter into results in sparse matrix multiplication

Inner Product
Matrix Multiplication

$$\forall_i \forall_j \forall_k \; A_{\underset{\rightarrow}{ij}} \mathrel{+}= B_{\underset{\rightarrow}{ik}} C_{\underset{\leftarrow}{kj}}$$

CSR

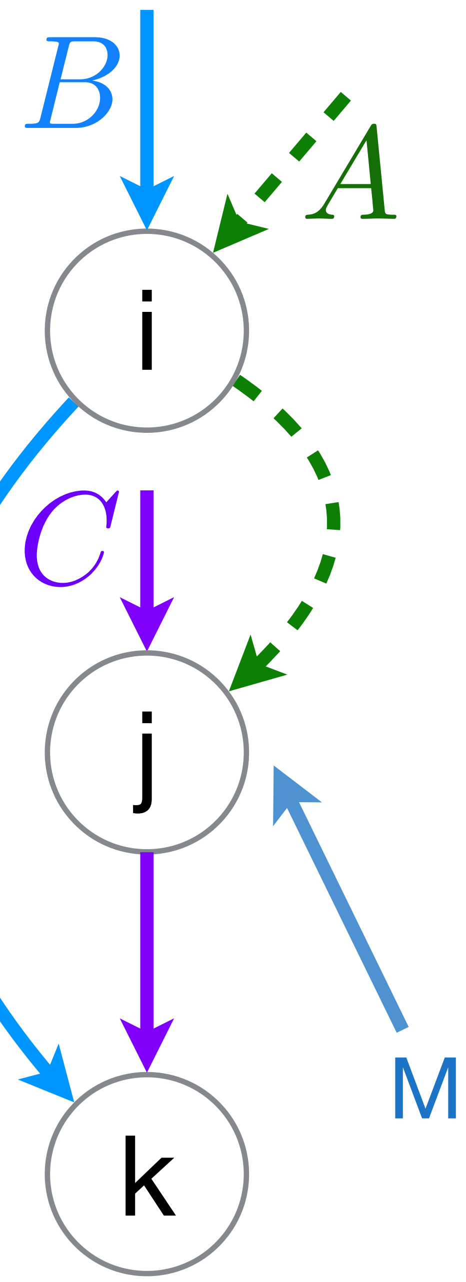
A	rows	Dense
	cols	Compressed

CSR

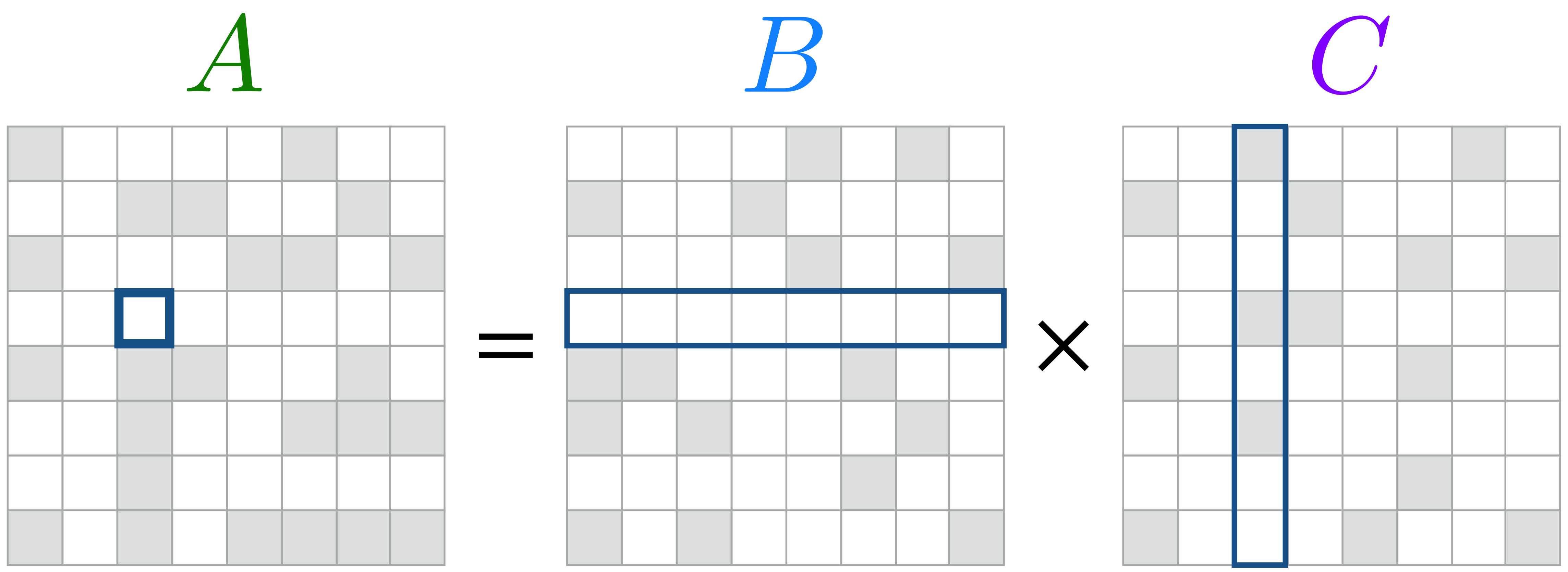
B	rows	Dense
	cols	Compressed

CSC

C	cols	Dense
	rows	Compressed



Must iterate over every i,j (inefficient)



Workspace to scatter into results in sparse matrix multiplication

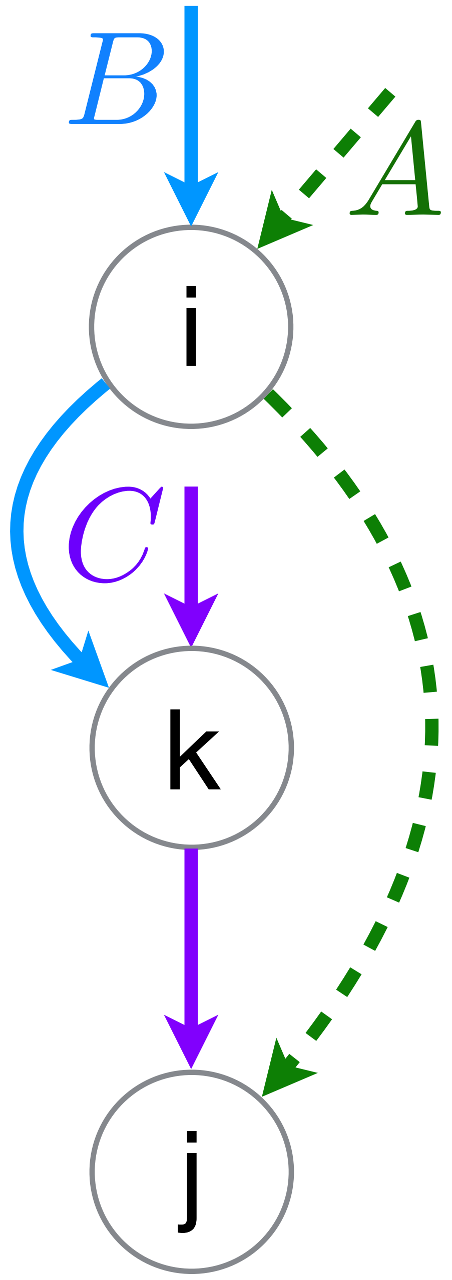
Linear Combination of Rows
Matrix Multiplication

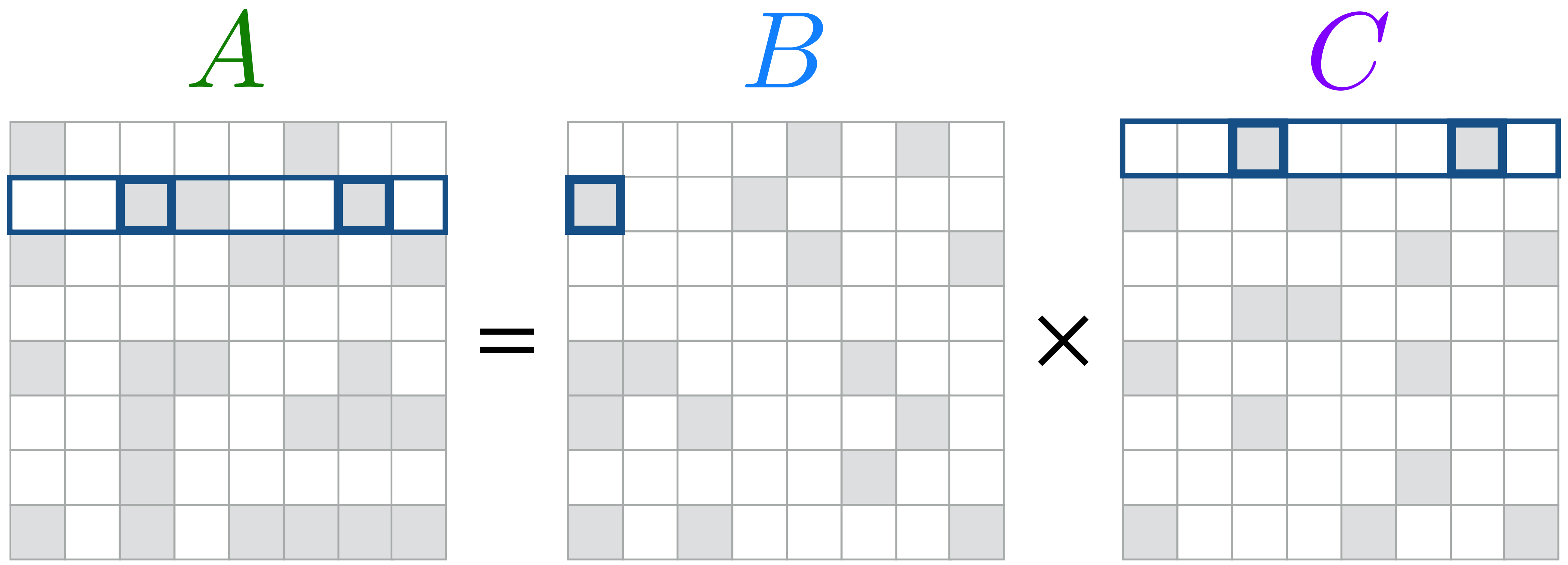
$$\forall_i \forall_k \forall_j \;\; \underrightarrow{A_{ij}} \mathrel{+}= \underrightarrow{B_{ik}}\,\underrightarrow{C_{kj}}$$

		CSR
A	rows	Dense
	cols	Compressed

		CSR
B	rows	Dense
	cols	Compressed

		CSR
C	rows	Dense
	cols	Compressed





Workspace to scatter into results in sparse matrix multiplication

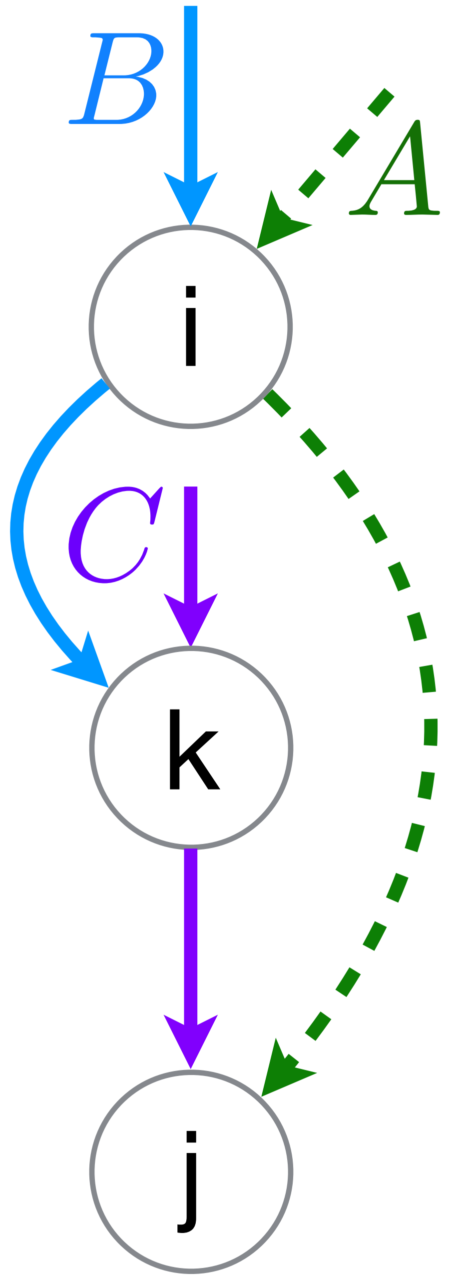
Linear Combination of Rows
Matrix Multiplication

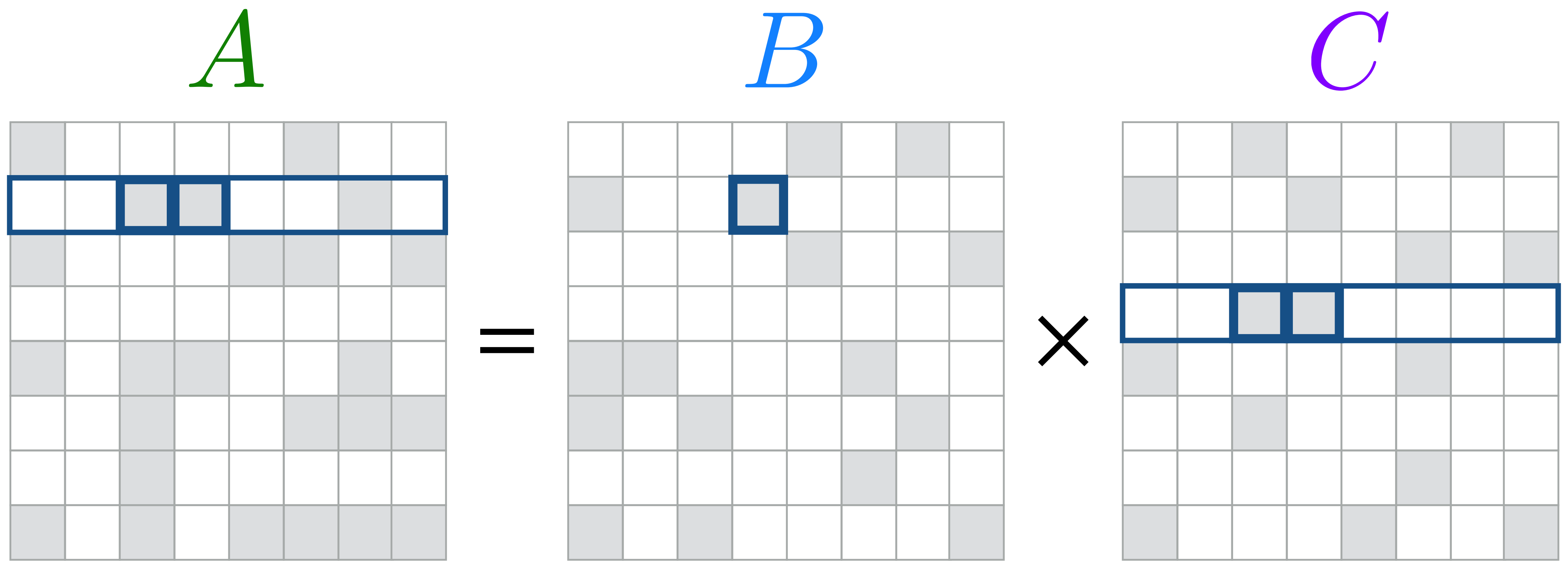
$$\forall_i \forall_k \forall_j \; A_{\underrightarrow{ij}} \mathrel{+}= B_{\underrightarrow{ik}} C_{\underrightarrow{kj}}$$

		CSR
A	rows	Dense
	cols	Compressed

		CSR
B	rows	Dense
	cols	Compressed

		CSR
C	rows	Dense
	cols	Compressed





Workspace to scatter into results in sparse matrix multiplication

Linear Combination of Rows
Matrix Multiplication

$$\forall_i \forall_k \forall_j \;\; A_{ij} \mathrel{+}= B_{ik} C_{kj}$$

CSR

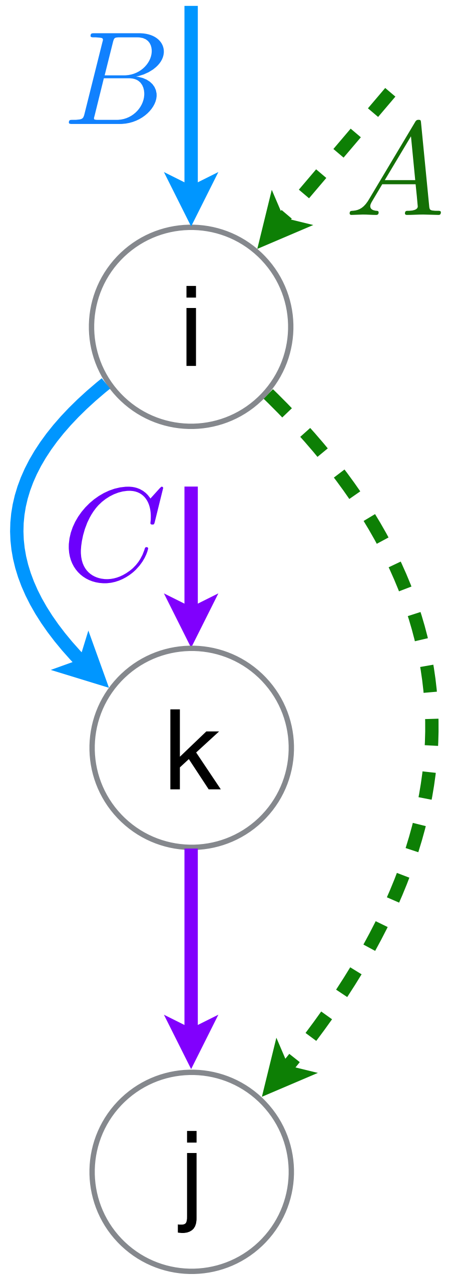
A		
	rows	Dense
	cols	Compressed

CSR

B		
	rows	Dense
	cols	Compressed

CSR

C		
	rows	Dense
	cols	Compressed



dense B_1

```
for (int i = 0; i < m; i++) {




}
```

Workspace to scatter into results in sparse matrix multiplication

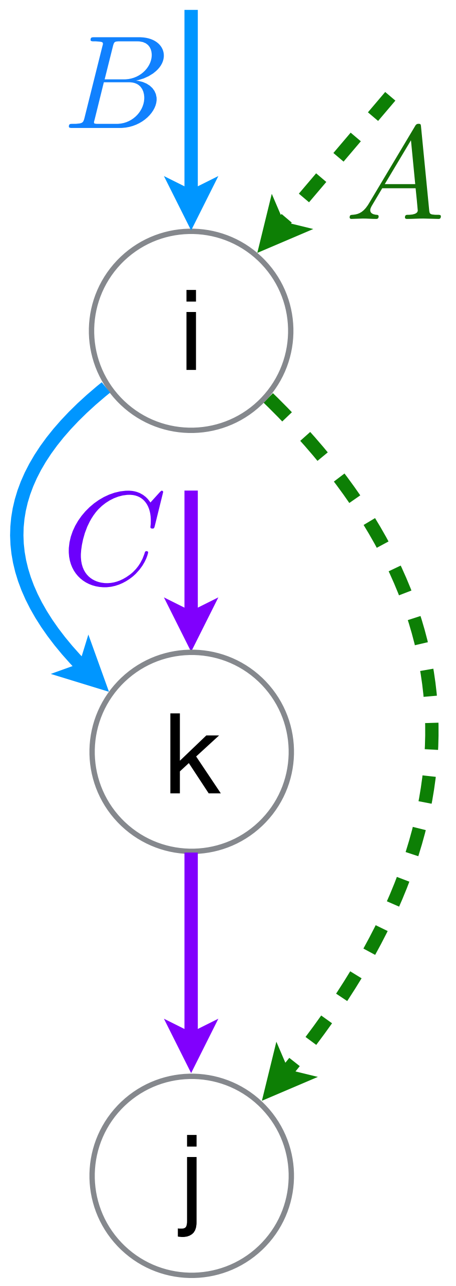
Linear Combination of Rows
Matrix Multiplication

$$\forall_i \forall_k \forall_j \; A_{ij} \mathrel{+}= B_{ik} C_{kj}$$

		CSR
A	rows	Dense
	cols	Compressed

		CSR
B	rows	Dense
	cols	Compressed

		CSR
C	rows	Dense
	cols	Compressed



compressed B_2
dense C_1 →

```
for (int i = 0; i < m; i++) {


  for (int pB2 = B2_pos[i]; pB2 < B2_pos[i+1]; pB2++) {
    int k = B2_crd[pB2];



  }


}
```

Workspace to scatter into results in sparse matrix multiplication

Linear Combination of Rows
Matrix Multiplication

$$\forall_i \forall_k \forall_j \; A_{ij} \mathrel{+}= B_{ik} C_{kj}$$

A

CSR

rows	Dense
cols	Compressed

B

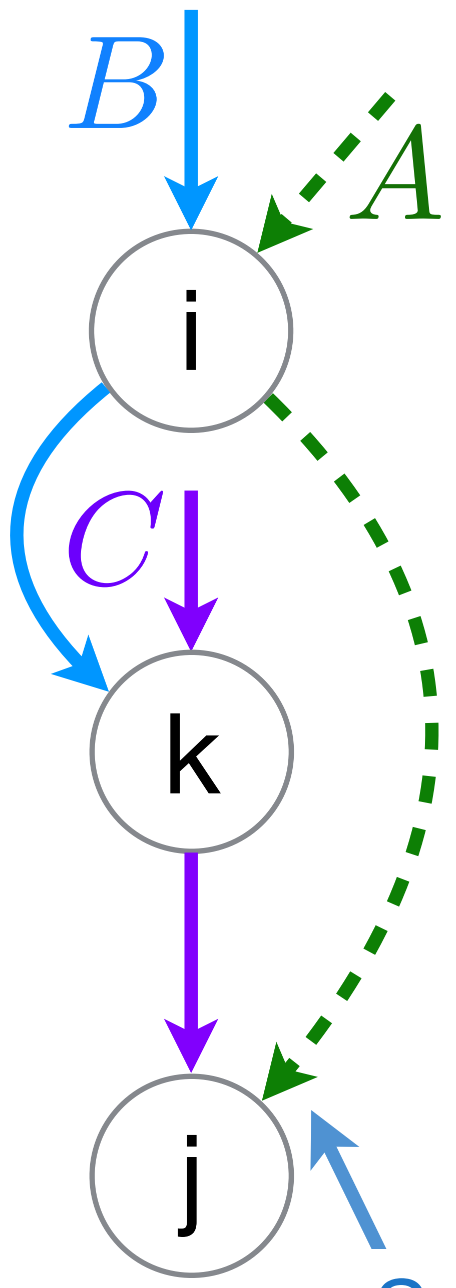
CSR

rows	Dense
cols	Compressed

C

CSR

rows	Dense
cols	Compressed



Scatter results into compressed A (inefficient)

```
for (int i = 0; i < m; i++) {


 for (int pB2 = B2_pos[i]; pB2 < B2_pos[i+1]; pB2++) {
  int k = B2_crd[pB2];



 }



}
```

Workspace to scatter into results in sparse matrix multiplication

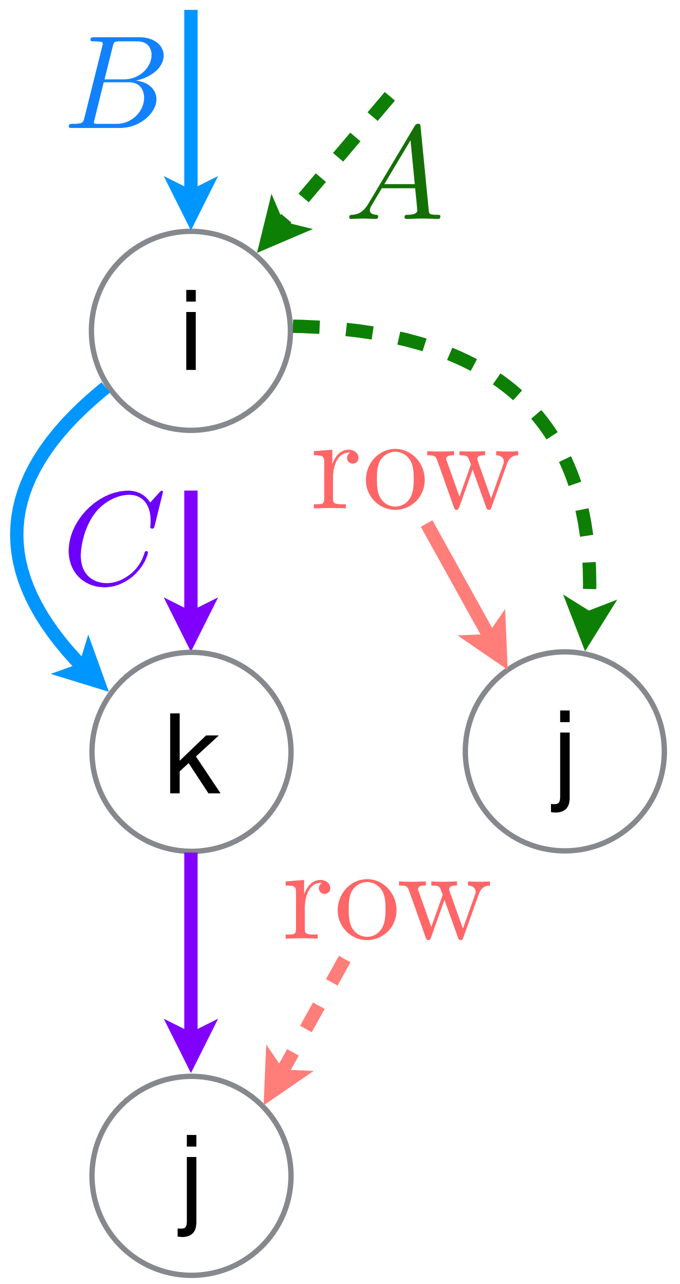
Linear Combination of Rows
Matrix Multiplication

$$\forall_i\left(\forall_j\ A_{ij} = \mathrm{row}_j\right)\ \textbf{where}\ \left(\forall_k\forall_j\ \mathrm{row}_j \mathrel{+}= B_{ik}C_{kj}\right)$$

		CSR
A	rows	Dense
	cols	Compressed

		CSR
B	rows	Dense
	cols	Compressed

		CSR
C	rows	Dense
	cols	Compressed



```
for (int i = 0; i < m; i++) {


  for (int pB2 = B2_pos[i]; pB2 < B2_pos[i+1]; pB2++) {
    int k = B2_crd[pB2];



  }



}
```

Workspace to scatter into results in sparse matrix multiplication

Linear Combination of Rows
Matrix Multiplication

$$\forall_i \big(\forall_j\ A_{ij} = \mathrm{row}_j\big)\ \textbf{where}\ \big(\forall_k \forall_j\ \mathrm{row}_j \mathrel{+}= B_{ik} C_{kj}\big)$$

CSR

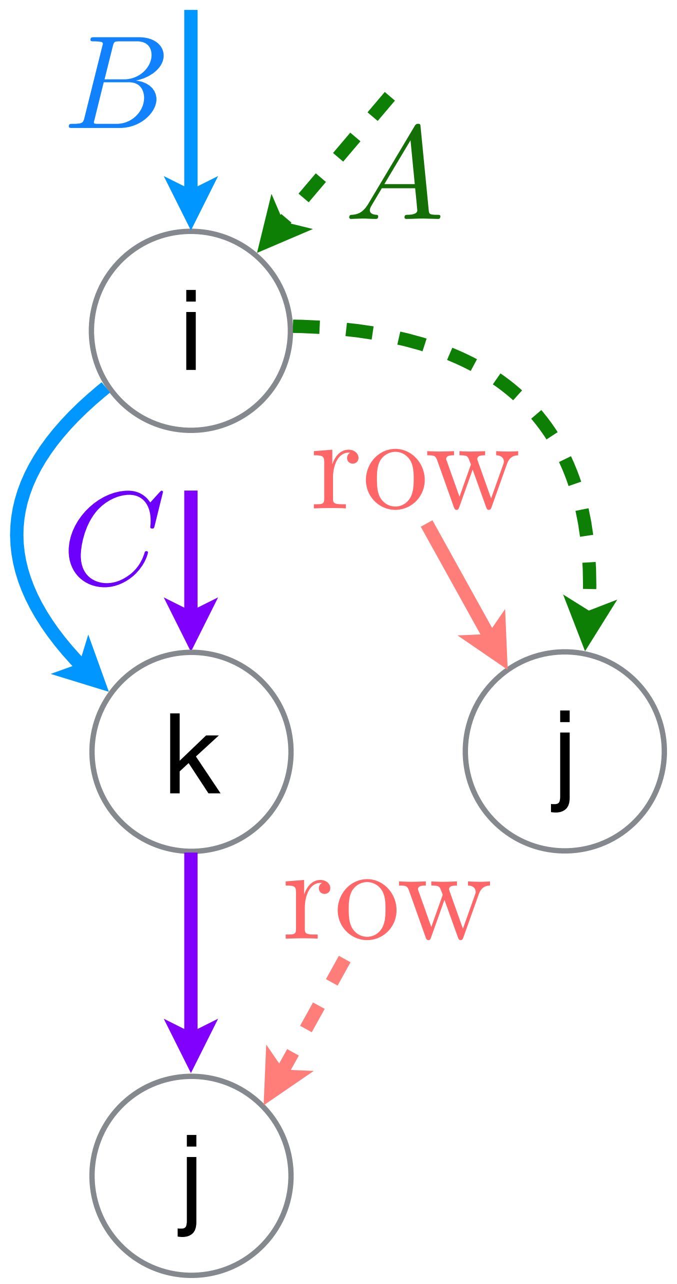
A	rows	Dense
	cols	Compressed

CSR

B	rows	Dense
	cols	Compressed

CSR

C	rows	Dense
	cols	Compressed



compressed C_2
dense row

```
for (int i = 0; i < m; i++) {



 for (int pB2 = B2_pos[i]; pB2 < B2_pos[i+1]; pB2++) {
  int k = B2_crd[pB2];


  for (int pC2 = C2_pos[k]; pC2 < C2_pos[k+1]; pC2++) {
   int j = C2_crd[pC2];
   row[j] += B[pB2] * C[pC2];
  }


 }




}
```

Workspace to scatter into results in sparse matrix multiplication

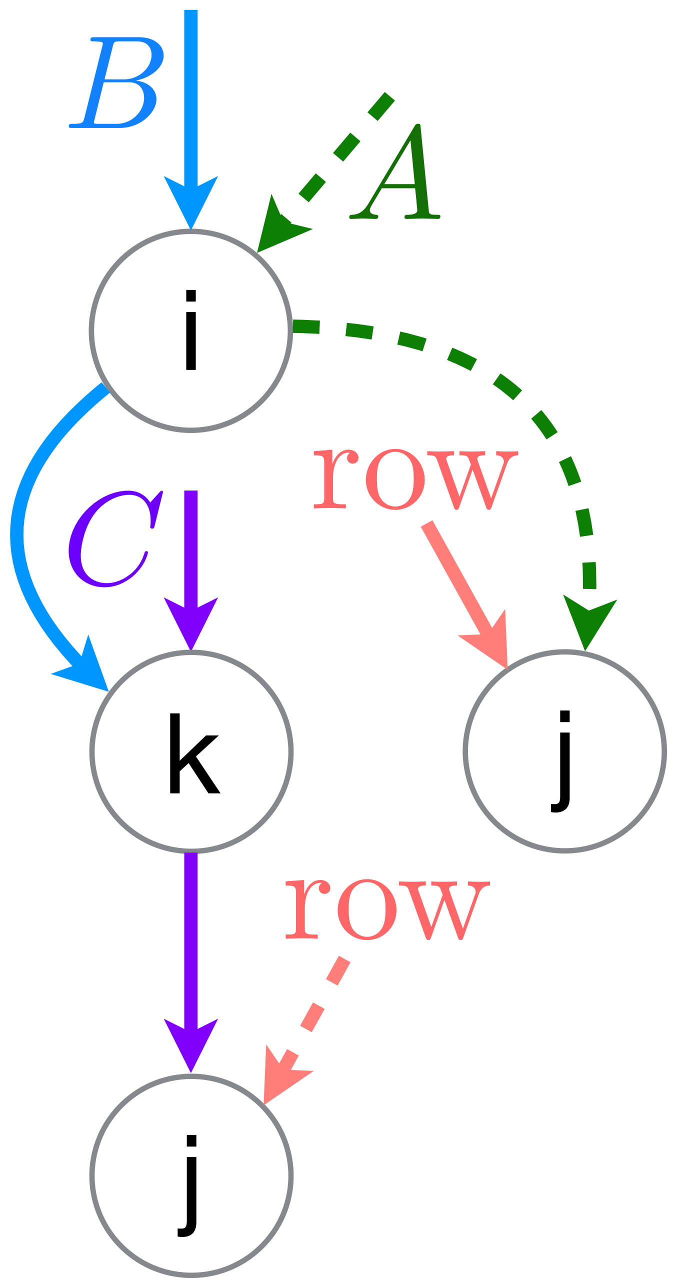
Linear Combination of Rows
Matrix Multiplication

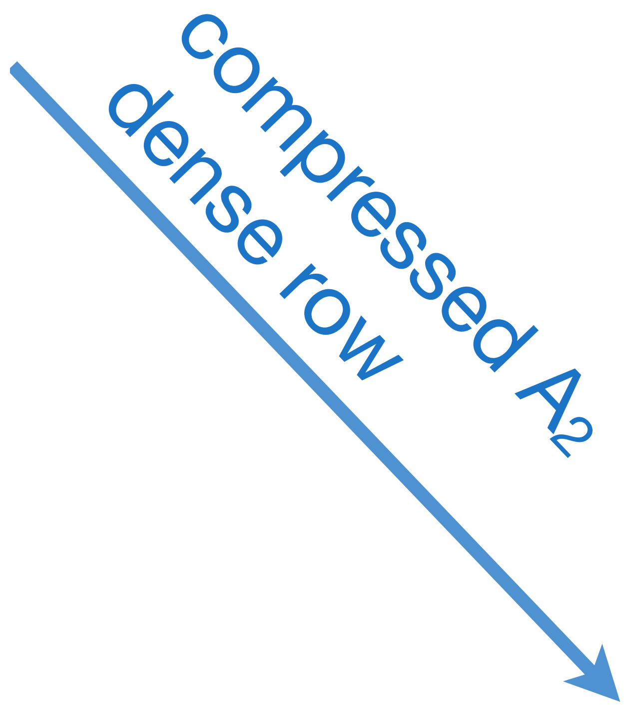
$$\forall_i \left(\forall_j \;\; A_{ij} = \mathrm{row}_j\right) \;\textbf{where}\; \left(\forall_k \forall_j \;\; \mathrm{row}_j \mathrel{+}= B_{ik} C_{kj}\right)$$

		CSR
A	rows	Dense
	cols	Compressed

		CSR
B	rows	Dense
	cols	Compressed

		CSR
C	rows	Dense
	cols	Compressed





```
for (int i = 0; i < m; i++) {



  for (int pB2 = B2_pos[i]; pB2 < B2_pos[i+1]; pB2++) {
    int k = B2_crd[pB2];



    for (int pC2 = C2_pos[k]; pC2 < C2_pos[k+1]; pC2++) {
      int j = C2_crd[pC2];
      row[j] += B[pB2] * C[pC2];
    }


  }

  for (int pA2 = A2_pos[i]; pA2 < A2_pos[i+1]; pA2++) {
    int j = A2_crd[pA2];
    A[pA2] = row[j];
    row[j] = 0.0;
  }

}
```

Workspace to scatter into results in sparse matrix multiplication

Linear Combination of Rows
Matrix Multiplication

$$\forall_i \big(\forall_j\ A_{ij} = \mathrm{row}_j\big)\ \textbf{where}\ \big(\forall_k \forall_j\ \mathrm{row}_j \mathrel{+}= B_{ik} C_{kj}\big)$$

CSR

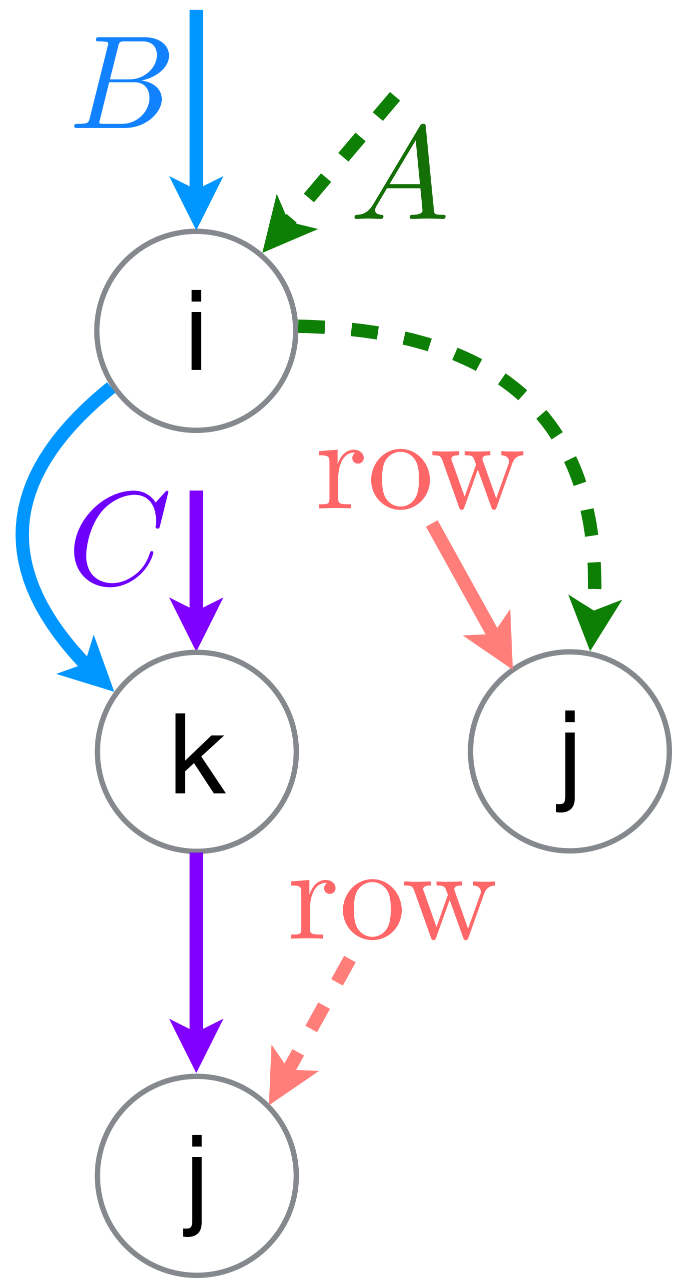
A		
	rows	Dense
	cols	Compressed

CSR

B		
	rows	Dense
	cols	Compressed

CSR

C		
	rows	Dense
	cols	Compressed

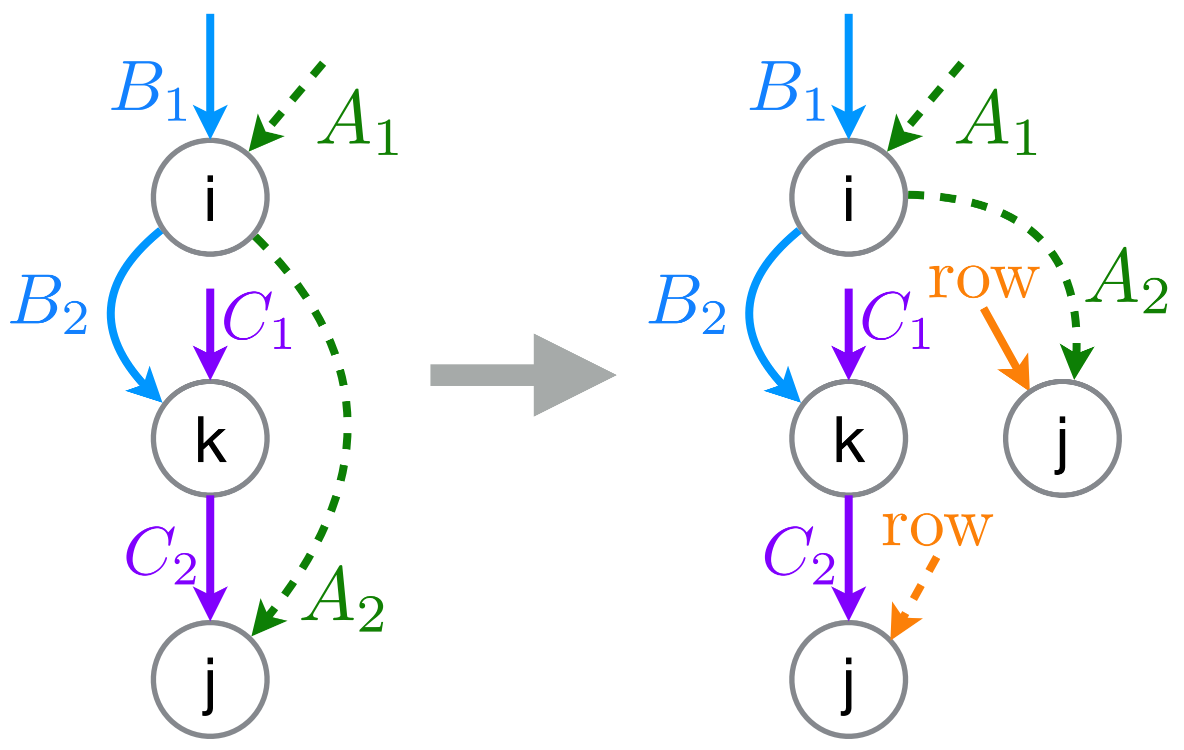


[Gustavson 1978]

```
for (int i = 0; i < m; i++) {



  for (int pB2 = B2_pos[i]; pB2 < B2_pos[i+1]; pB2++) {
    int k = B2_crd[pB2];



    for (int pC2 = C2_pos[k]; pC2 < C2_pos[k+1]; pC2++) {
      int j = C2_crd[pC2];
      row[j] += B[pB2] * C[pC2];
    }


  }

  for (int pA2 = A2_pos[i]; pA2 < A2_pos[i+1]; pA2++) {
    int j = A2_crd[pA2];
    A[pA2] = row[j];
    row[j] = 0.0;
  }

}
```

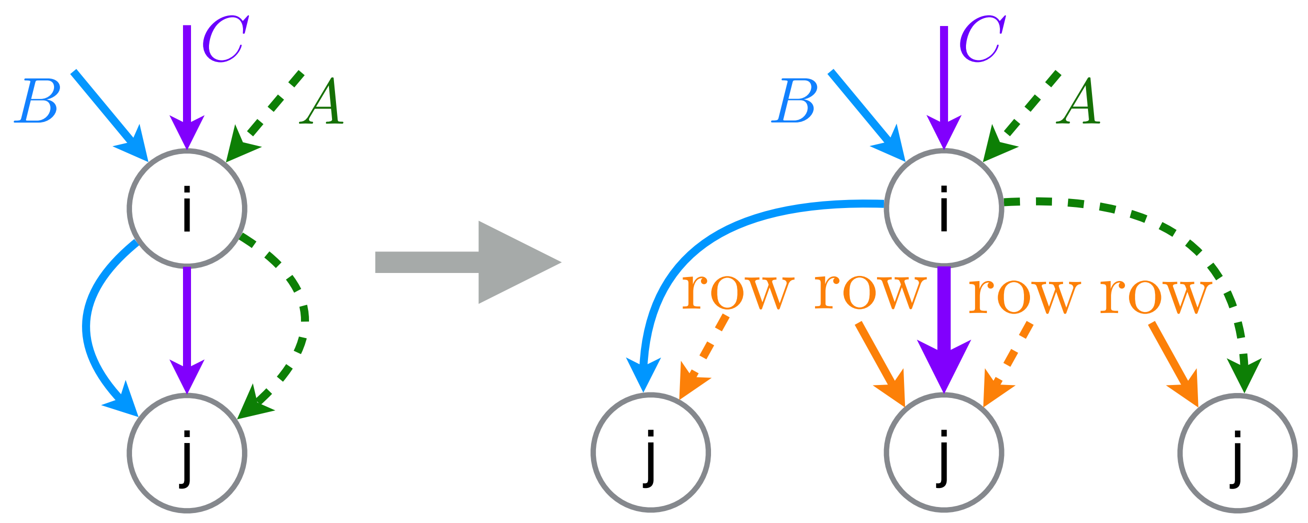
Other uses of where clauses

Scatter into results



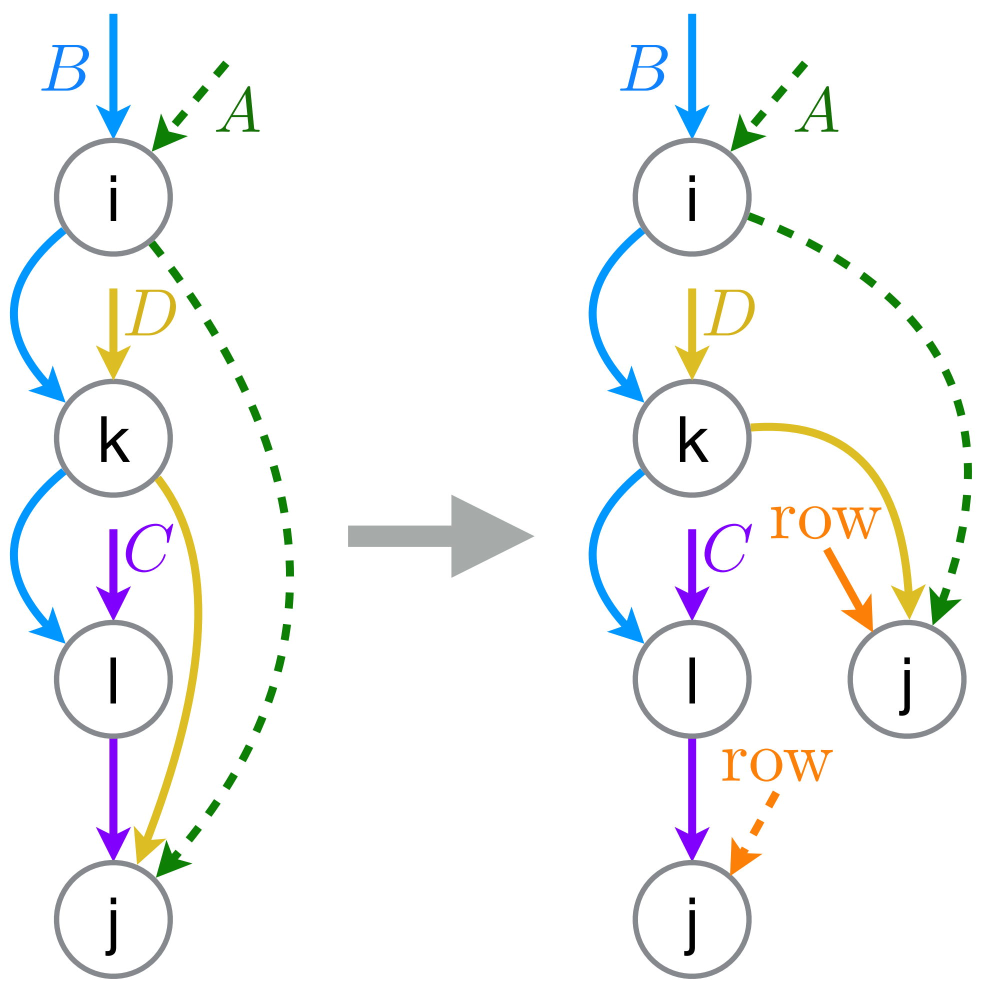
SpGEMM

Simplify merge code



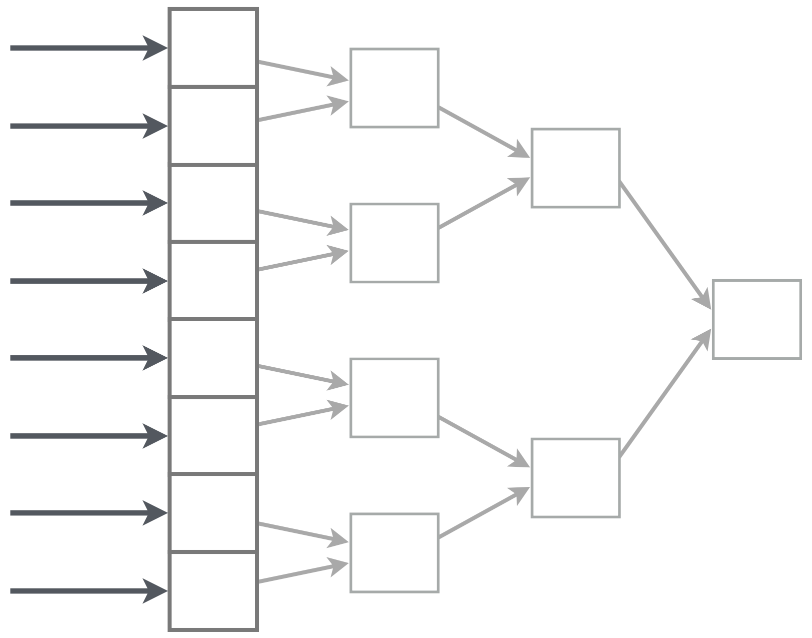
SpAdd

Loop-invariant code motion



MTTKRP

Prepare reductions



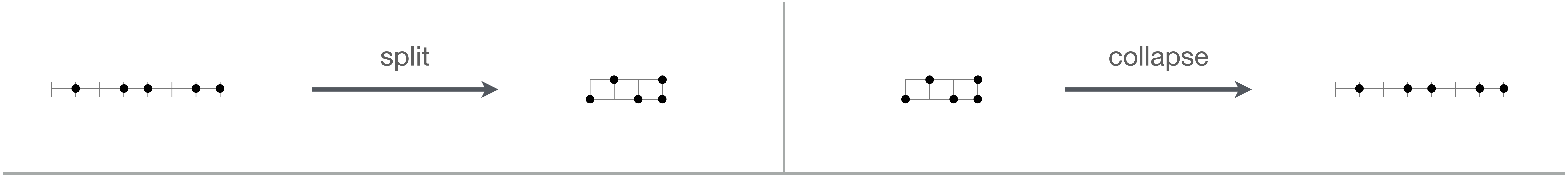
GPU shared memories

Mixed precision

```
for (int i = 0; i < m; i++) {
  double tj = 0.0;
  for (int pB2 = B2_pos[i];
           pB2 < B2_pos[i+1];
           pB2++) {
    int j = B2_crd[pB2];
    tj += B[pB2] * c[j];
  }
  a[i] = tj;
}
```

Single-precision floating point

A derived iteration space is one where dimensions have been split or collapsed



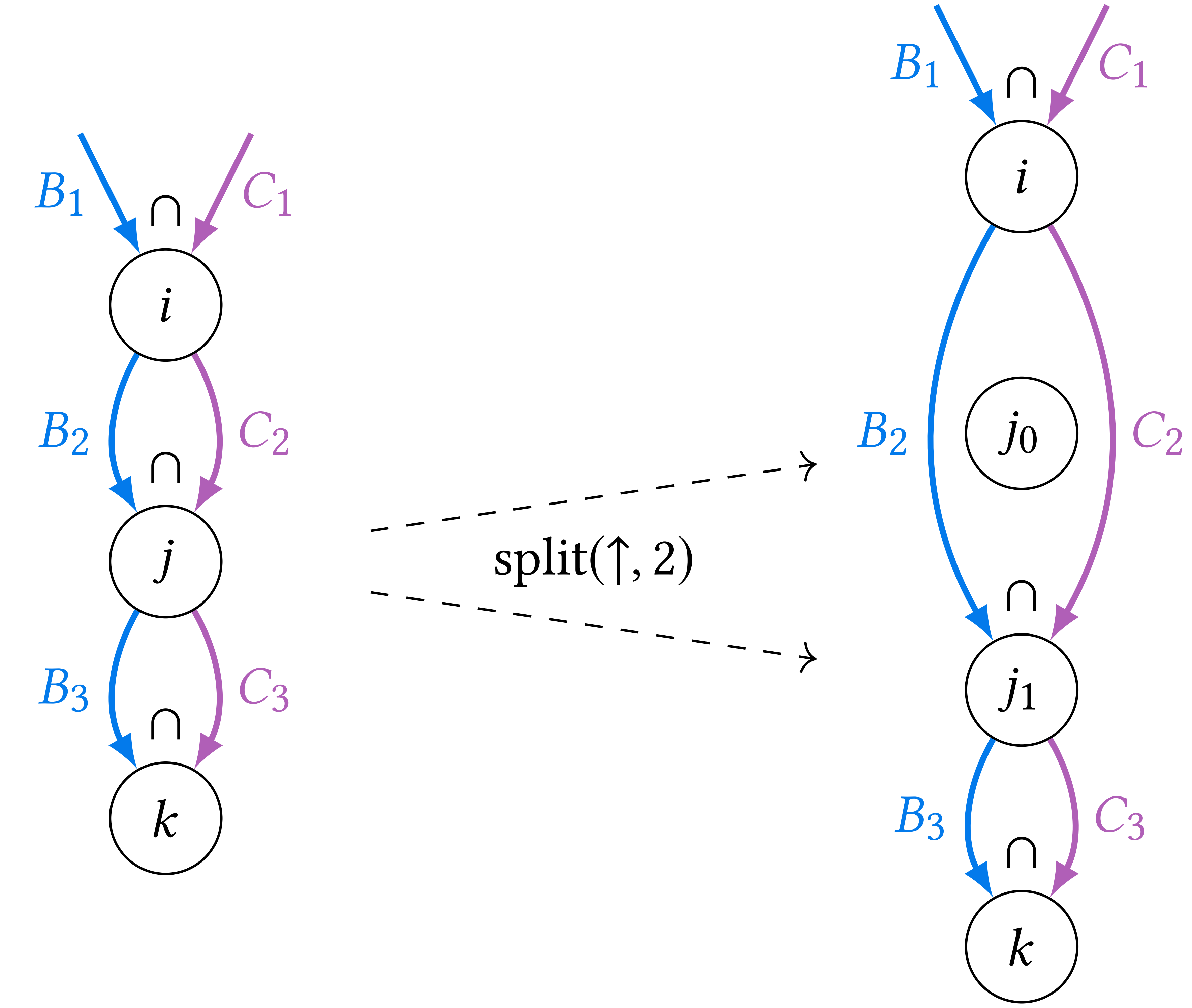
$$\forall_i\, a_i = b_i + c_i \xrightarrow{\text{split}} \forall_{i_0}\forall_{i_1}\, a_i = b_i + c_i \;:\; i \xrightarrow{\text{split}(\uparrow,4)} i_0 i_1$$

derived index variables

original index variable i

Cannot simply rewrite access expressions as with dense, so the mapping functions must be stored in the IR, so we can later emit code to remap coordinates.

The split iteration space function



$$\frac{\forall_i \forall_j \forall_k \, B_{ijk} \cap C_{ijk}}{\begin{array}{c} i \in B_1 \cap C_1 \\ j \in B_2 \cap C_2 \\ k \in B_3 \cap C_3 \end{array}}$$

$$\frac{\forall_i \forall_{j_0} \forall_{j_1} \forall_k \, B_{ijk} \cap C_{ijk} \; : \; j \xrightarrow{\text{split}(\uparrow,2)} j_0 j_1}{\begin{array}{l} i \in B_1 \cap C_1 \\ j_0 \in [0, 2) \\ j_1 \in B_2 \cap C_2 \\ k \in B_3 \cap C_3 \end{array}}$$

The split iteration space function comes in several variants

$$\mathrm{split}(\underline{\mathrm{direction}, \mathrm{stride}}, [\underline{\mathrm{tensor\ operand}}])$$

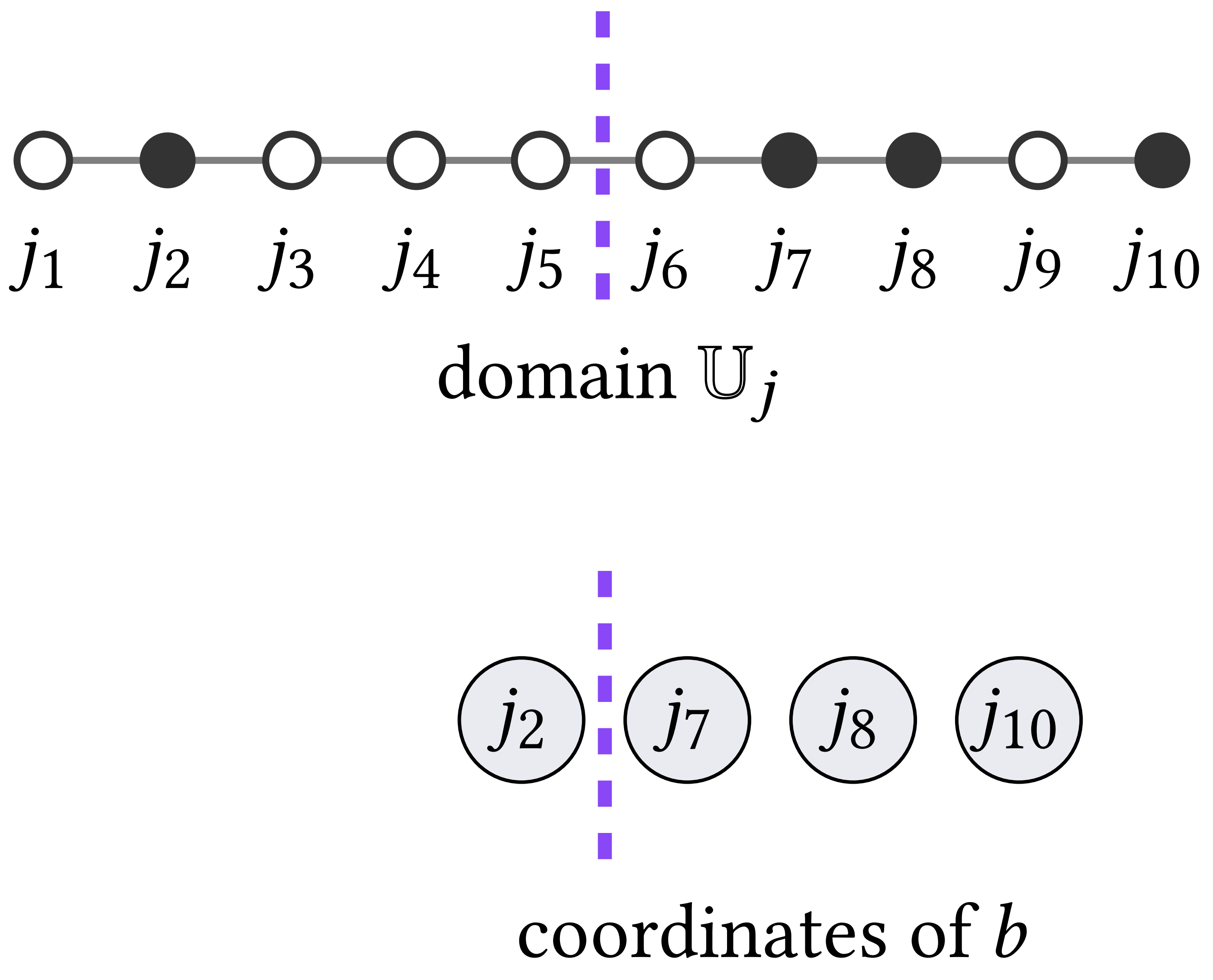
Split off a loop with stride iterations as the:

- outer loop (direction = ↑), or
- inner loop (direction = ↓)

Optional tensor operand whose nonzero coordinates the split applies to. If not given, the split applies to the universe of coordinates of the split index variable.

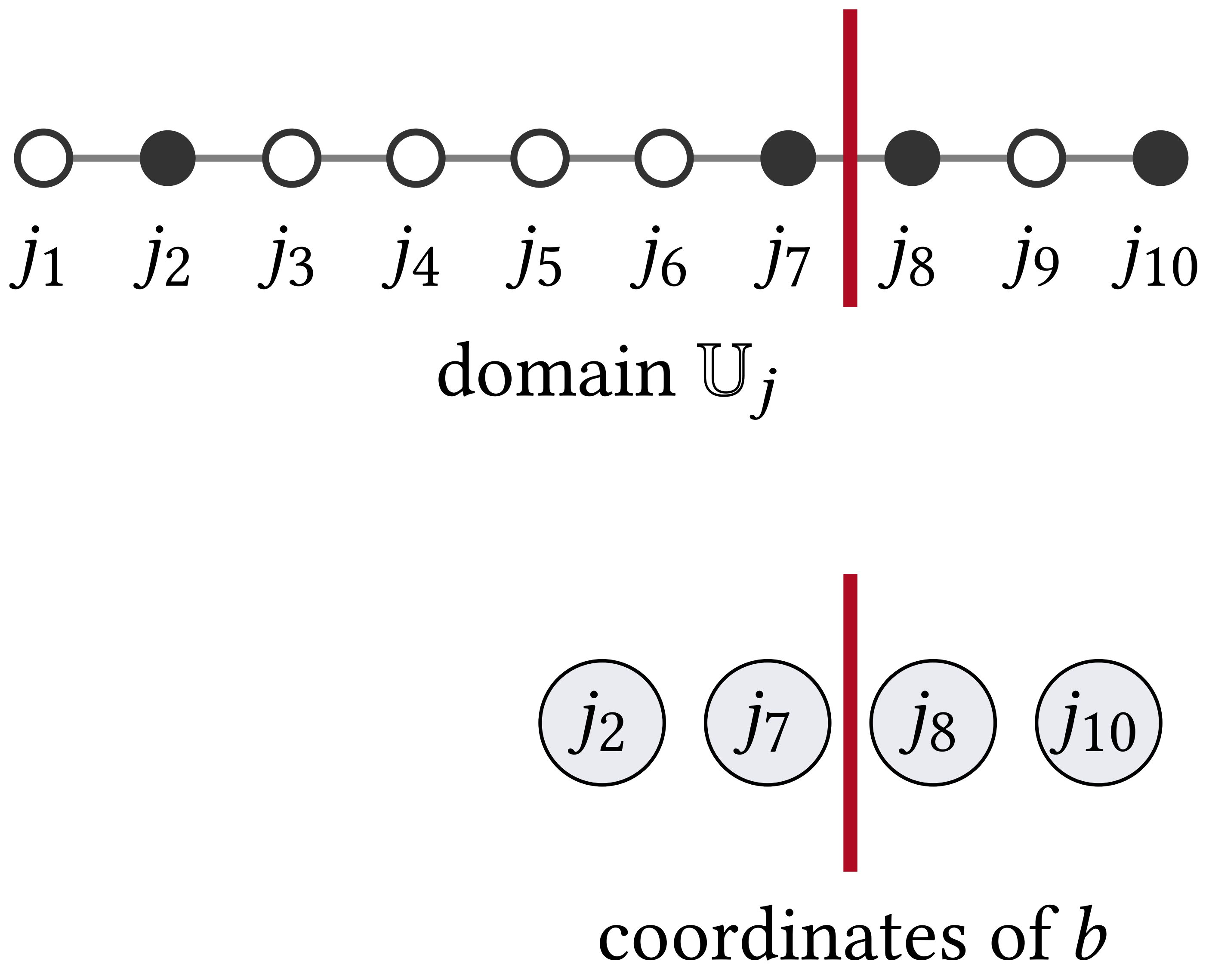
The split iteration space function can split with respect to the universe of coordinates or the coordinates of one operand

Universe split



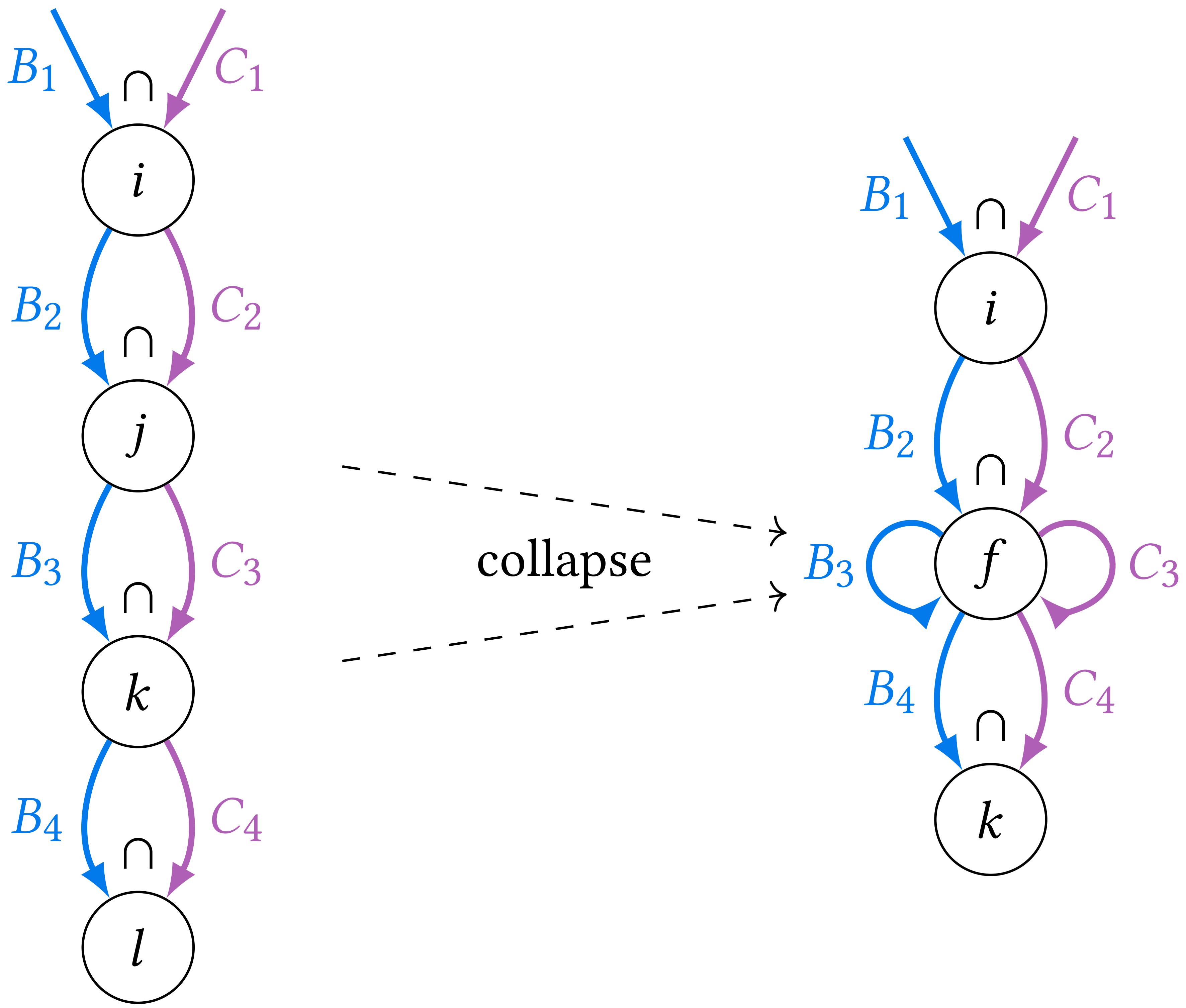
even split in coordinate universe, but
uneven split in b’s nonzero coordinates

Coordinate tree split



uneven split in coordinate universe, but
even split in b’s nonzero coordinates

The collapse iteration space function



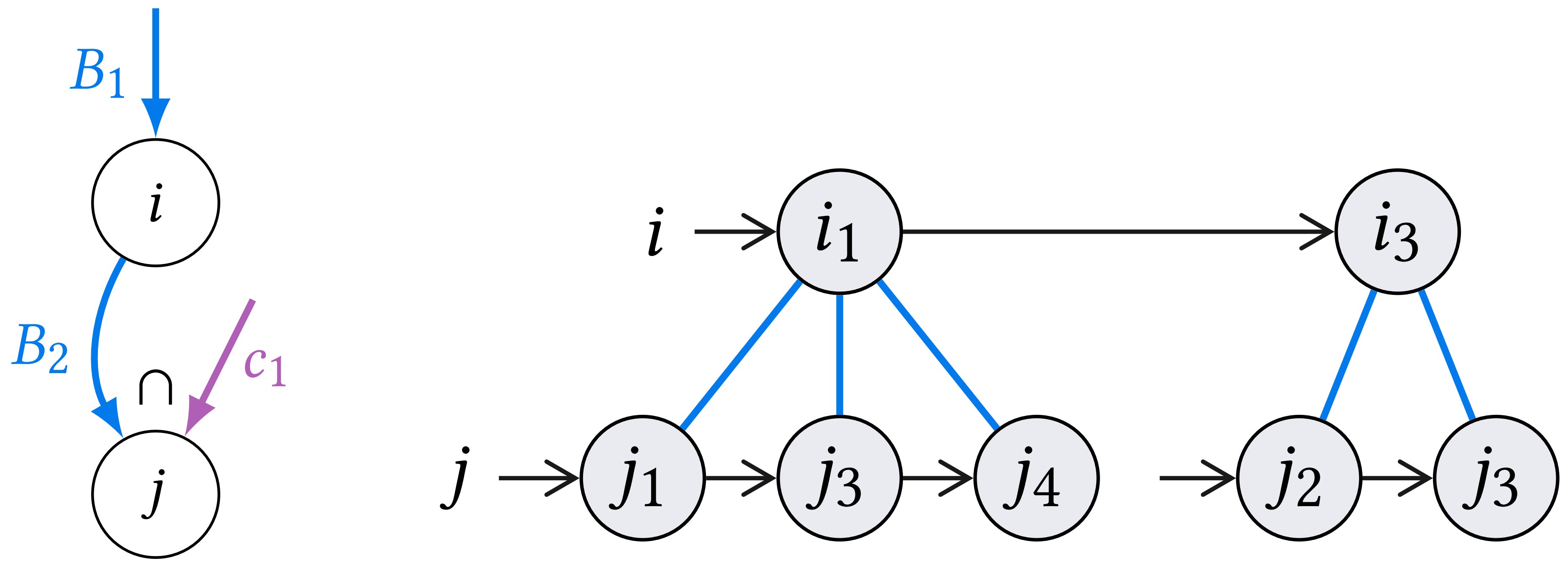
$$\frac{\forall_i \forall_j \forall_k \forall_l \; B_{ijkl} \cap C_{ijkl}}{\begin{array}{c} i \in B_1 \cap C_1 \\ j \in B_2 \cap C_2 \\ k \in B_3 \cap C_3 \\ l \in B_4 \cap C_4 \end{array}}$$

$$\frac{\forall_i \forall_f \forall_l \; B_{ijkl} \cap C_{ijkl} \; : \; jk \xrightarrow{\text{collapse}} f}{\begin{array}{l} i \in B_1 \cap C_1 \\ f \in (B_2 \times B_3) \cap (C_2 \times C_3) \\ k \in B_4 \cap C_4 \end{array}}$$

Iterate over Cartesian combination of coordinates in i and j.

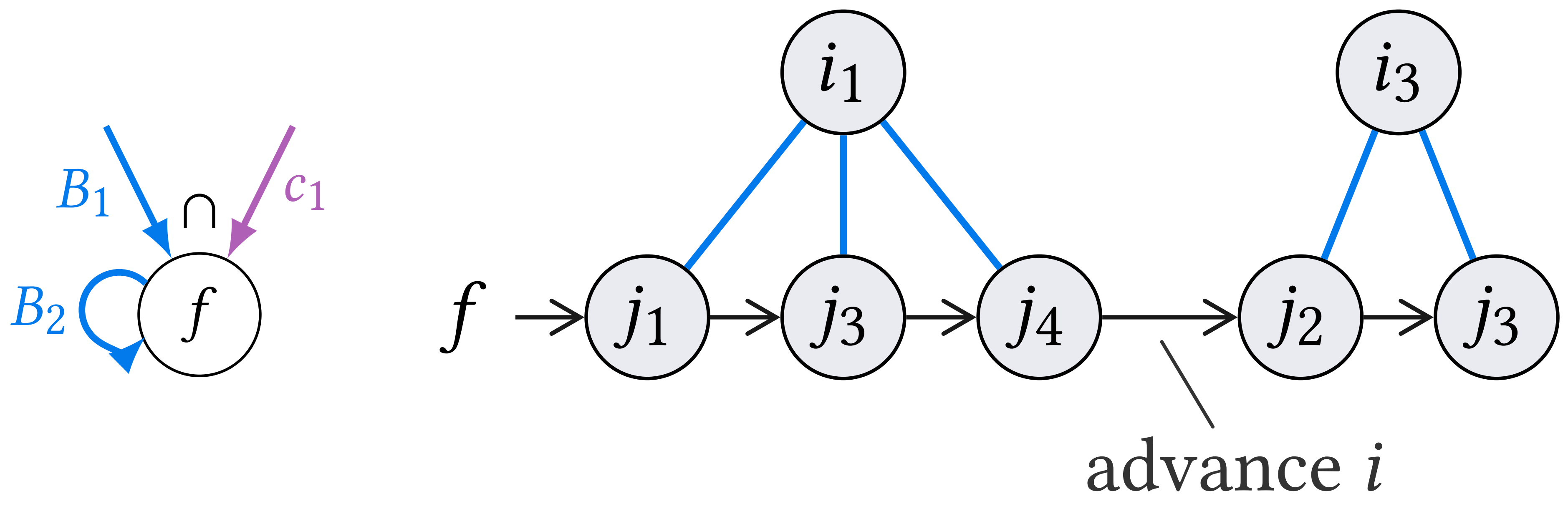
The collapse function leads to bottom-up iteration

Pre-collapse top-down iteration



```
for (int i = 0; i < m; i++) {
  for (int jB = B2_pos[i];
           jB < B2_pos[i+1]; jB++) {
    int j = B2_crd[jB];
    a[i] += B[jB] ∗ c[j];
  }
}
```

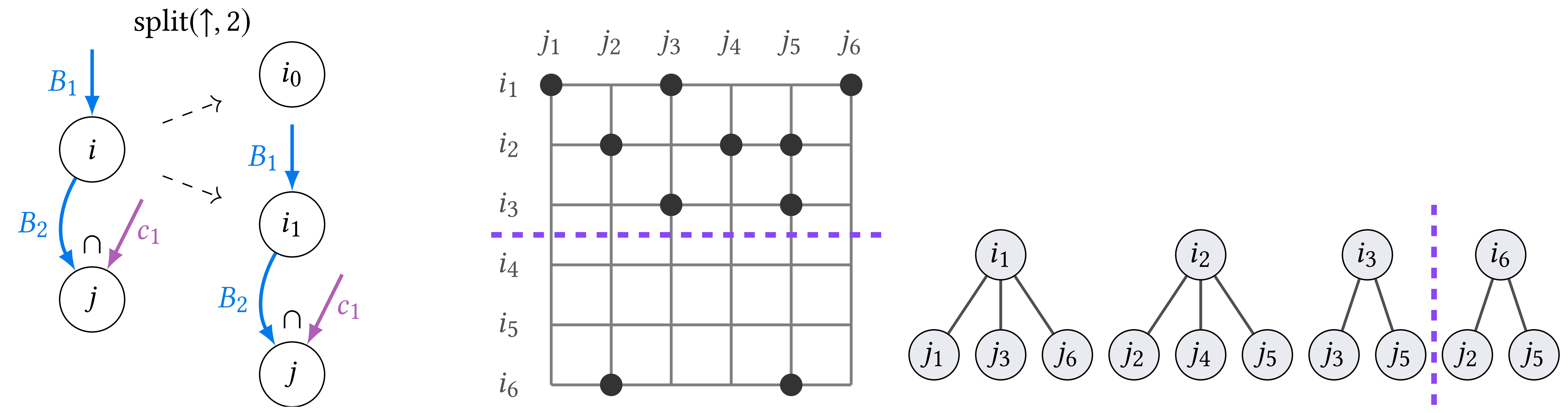
Post-collapse bottom-up iteration



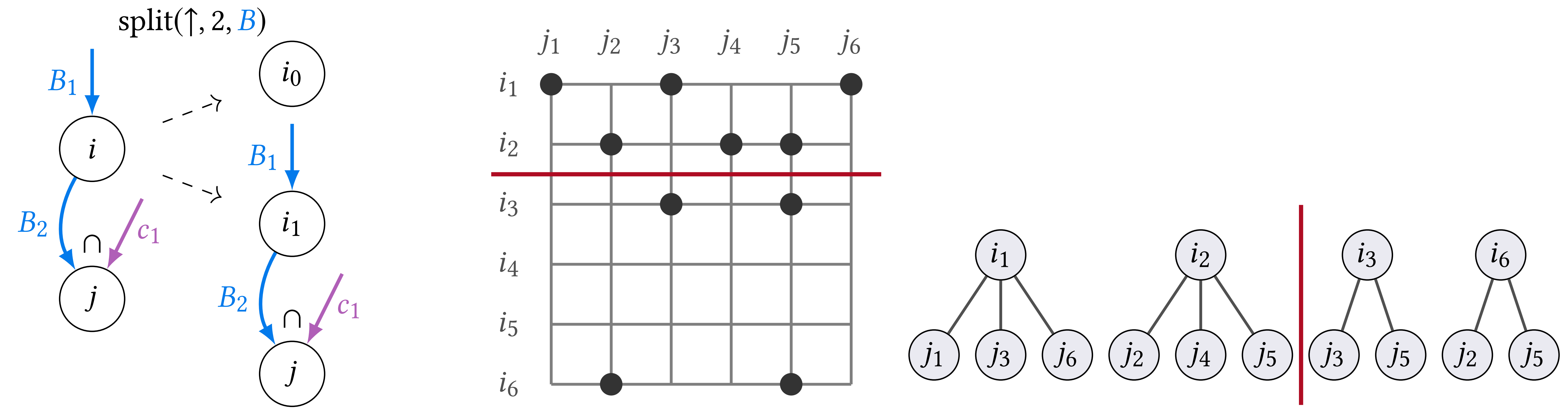
```
for (int f = 0, i = 0;
         f < B2_pos[m]; f++) {
  if (f >= B2_pos[m]) break;

  int j = B2_crd[f];
  while (f == B2_pos[i+1]) i++;

  a[i] += B[f] ∗ c[j];
}
```

Two-dimensional tiling examples

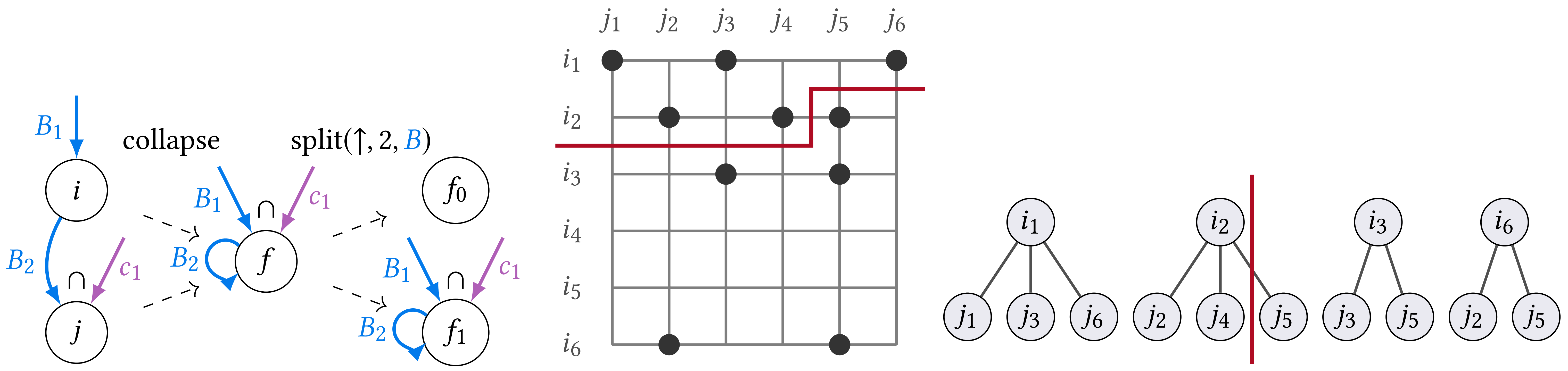
Universe split



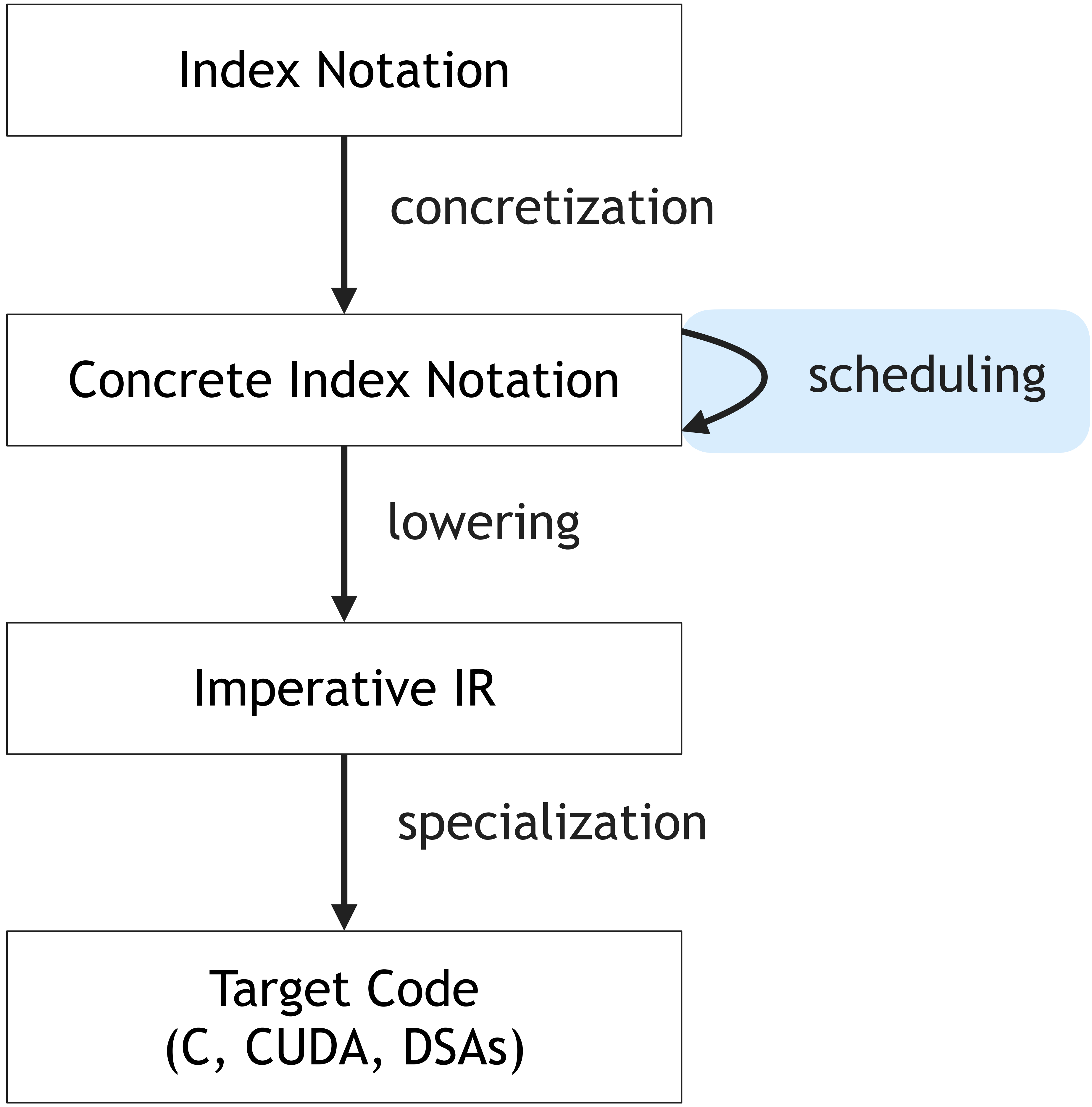
Coordinate tree split



Collapse+coordinate tree split



A sparse (tensor algebra) scheduling language



Index Notation

concretization

Concrete Index Notation

scheduling

lowering

Imperative IR

specialization

Target Code
(C, CUDA, DSAs)

- **reorder(i, j)** interchanges loops i and j
- **split(i, i_1, i_2, d, s, t)** strip-mines i into two loops i_1 and i_2, where i_1 or i_1 is of size s depending on the direction d. The tensor t is optional and, if given, means the loop is strip-mined w.r.t. its nonzeros.
- **collapse(i, j, f)** collapses loops i and j into a new loop f, which iterates over their Cartesian combination.
- **precompute(S, e, t, I)** precomputes expression e in index statement S before the loops I and stores the results in tensor t.
- **unroll**, **parallelize**, **vectorize**, …

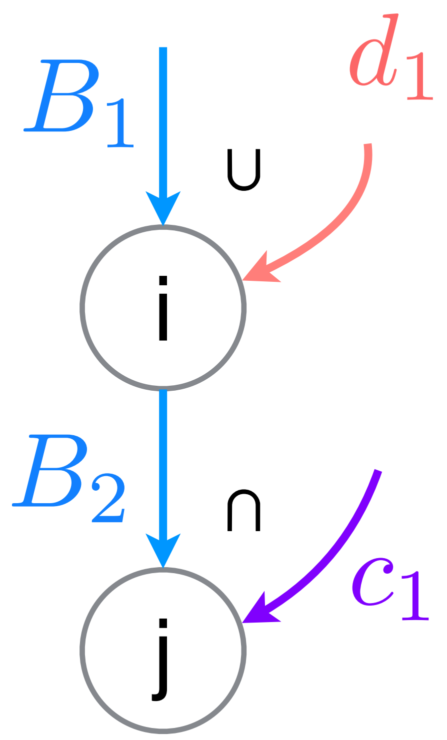
Lowering algorithm for code generation

```
function LOWER(assignment statement $S_{\text{assignment}}$)
    Emit compute code
end function

function LOWER(where statement $S_{\text{where}}$)
    LOWER(producer statement of $S_{\text{where}}$)
    LOWER(consumer statement of $S_{\text{where}}$)
end function

function LOWER(sequence statement $S_{\text{sequence}}$)
    LOWER(definition statement of $S_{\text{sequence}}$)
    LOWER(mutation statement of $S_{\text{sequence}}$)
end function

function LOWER(multi statement $S_{\text{multi}}$)
    LOWER(left statement of $S_{\text{multi}}$)
    LOWER(right statement of $S_{\text{multi}}$)
end function
```

```
function LOWER(forall statement $S_{\text{forall}}$ of index variable $i$)
    let $\mathcal{L}$ be an iteration lattice constructed from $S_{\text{forall}}$
    Emit initialize iterators
    for each lattice point $\mathcal{L}_p$ in $\mathcal{L}$ do
        Emit loop header
        Emit access iterators
        Emit map candidate coordinates to the original space
        Emit resolve the coordinate of $i$
        Emit map resolved coordinate to each derived space
        Emit locate from locators
        for each lattice point $\mathcal{L}_q < \mathcal{L}_p$ in $\mathcal{L}$ do
            Emit conditional header
            let $S_{\text{simplified}}$ be a statement constructed from
                the body of $S_{\text{forall}}$ by removing operands
                that have run out of values in $\mathcal{L}_q$
            LOWER($S_{\text{simplified}}$)
            Emit assembly code
            Emit conditional footer
        end for
        Emit advance iterators
        Emit loop footer
    end for
end function
```

Recursive lowering of concrete index notation

$$a_i = B_{ij} c_j + d_i$$



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function LOWER(forall statement $S_{\text{forall}}$ of index variable $i$)
    let $\mathcal{L}$ be an iteration lattice constructed from $S_{\text{forall}}$
    Emit initialize iterators
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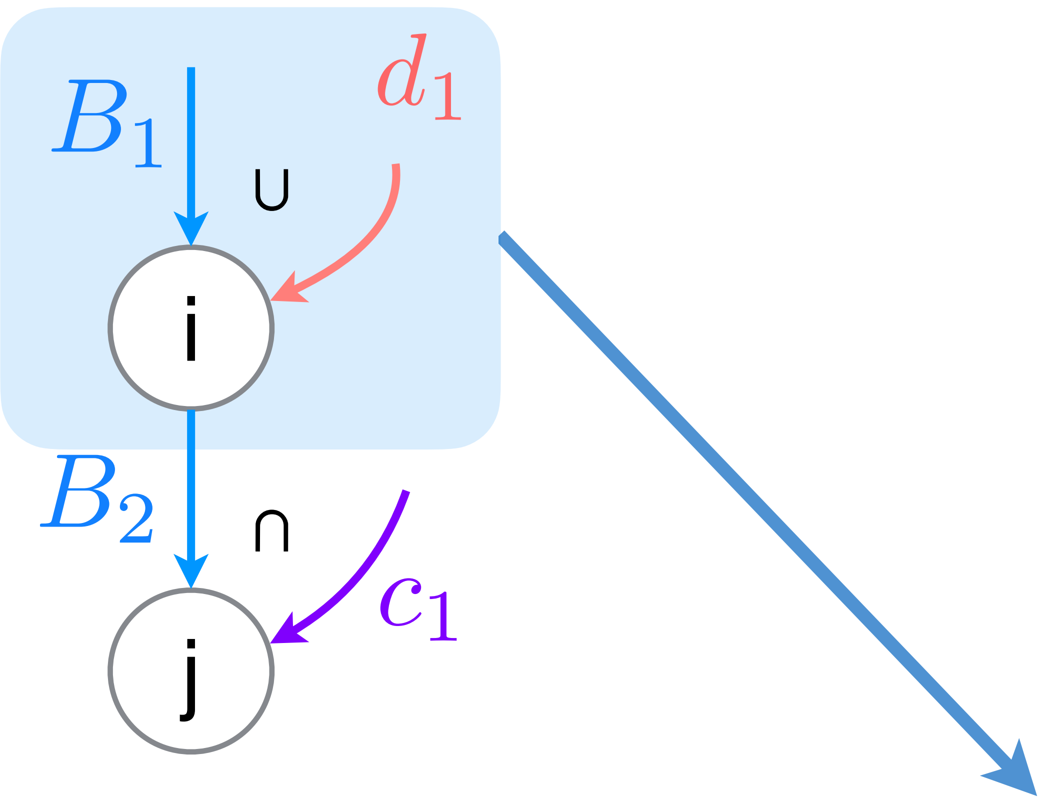
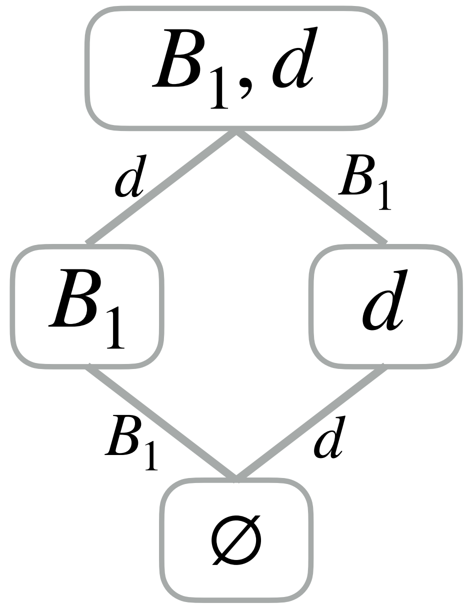
        Emit loop header
        Emit access iterators
        Emit resolve the coordinate of $i$
        Emit locate from locators
        for each lattice point $\mathcal{L}_q < \mathcal{L}_p$ in $\mathcal{L}$ do
            Emit conditional header
            let $S_{\text{simplified}}$ be a statement constructed from
                the body of $S_{\text{forall}}$ by removing operands
                that have run out of values in $\mathcal{L}_q$
            LOWER($S_{\text{simplified}}$)
            Emit conditional footer
        end for
        Emit advance iterators
        Emit loop footer
    end for
end function
```

Recursive lowering of concrete index notation

$$a_i = B_{ij}\, c_j + d_i$$





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function LOWER(forall statement $S_{\text{forall}}$ of index variable $i$)
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            LOWER($S_{\text{simplified}}$)
            Emit conditional footer
        end for
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        Emit loop footer
    end for
end function
```
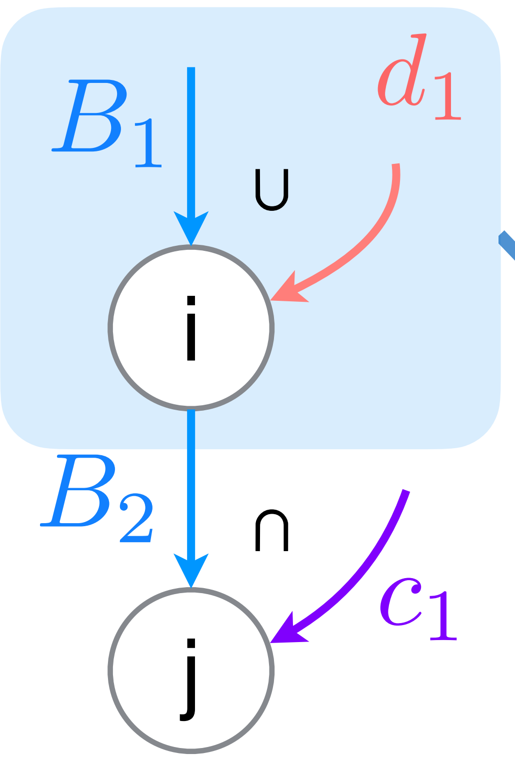

Recursive lowering of concrete index notation

$$a_i = B_{ij} c_j + d_i$$



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function LOWER(forall statement $S_{\text{forall}}$ of index variable $i$)
    let $\mathcal{L}$ be an iteration lattice constructed from $S_{\text{forall}}$
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        Emit resolve the coordinate of $i$
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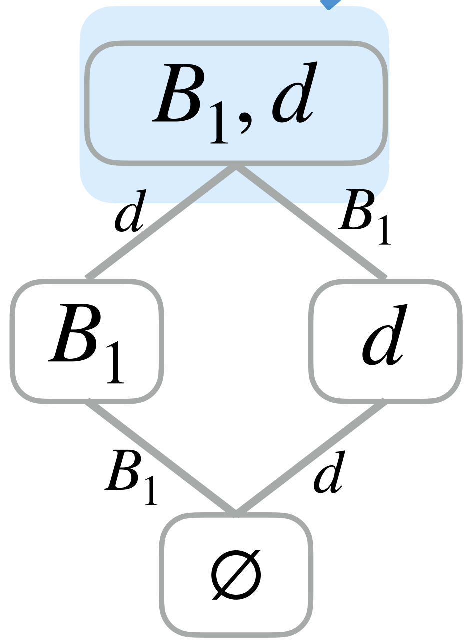
                that have run out of values in $\mathcal{L}_q$
            LOWER($S_{\text{simplified}}$)
            Emit conditional footer
        end for
        Emit advance iterators
        Emit loop footer
    end for
end function
```

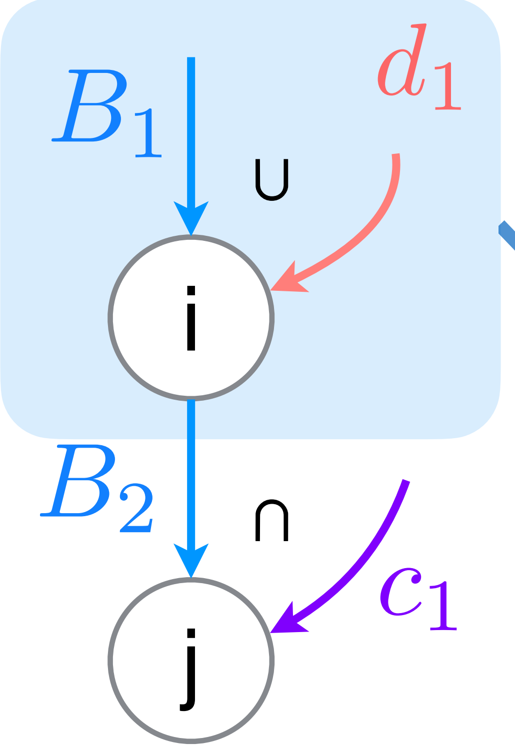


```
int iB = B1_pos[0];
int id = d1_pos[0];
while (iB < B1_pos[1] && id < d1_pos) {
  int iB = B1_crd[iB];
  int id = d1_crd[id];
  int i = min(iB, id);


  if (iB == i) iB++;
  if (id == i) id++;
}
```

Recursive lowering of concrete index notation

$$a_i = B_{ij}\,c_j + d_i$$



```
function LOWER(forall statement $S_{forall}$ of index variable $i$)
    let $\mathcal{L}$ be an iteration lattice constructed from $S_{forall}$
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        Emit locate from locators
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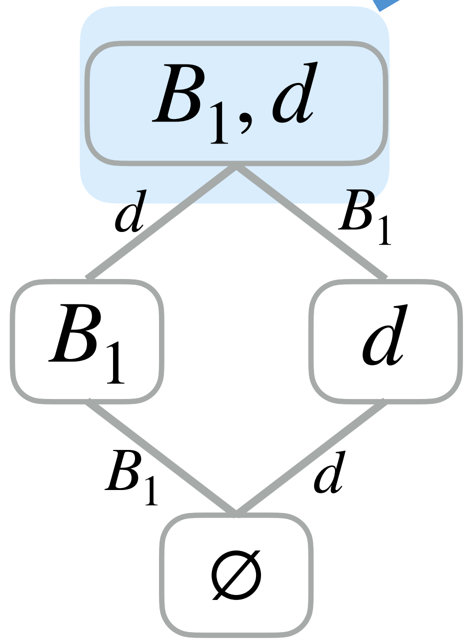
                the body of $S_{forall}$ by removing operands
                that have run out of values in $\mathcal{L}_q$
            LOWER($S_{simplified}$)
            Emit conditional footer
        end for
        Emit advance iterators
        Emit loop footer
    end for
end function
```



```
int iB = B1_pos[0];
int id = d1_pos[0];
while (iB < B1_pos[1] && id < d1_pos) {
  int iB = B1_crd[iB];
  int id = d1_crd[id];
  int i = min(iB, id);
  if (iB == i && id == i) {







  }





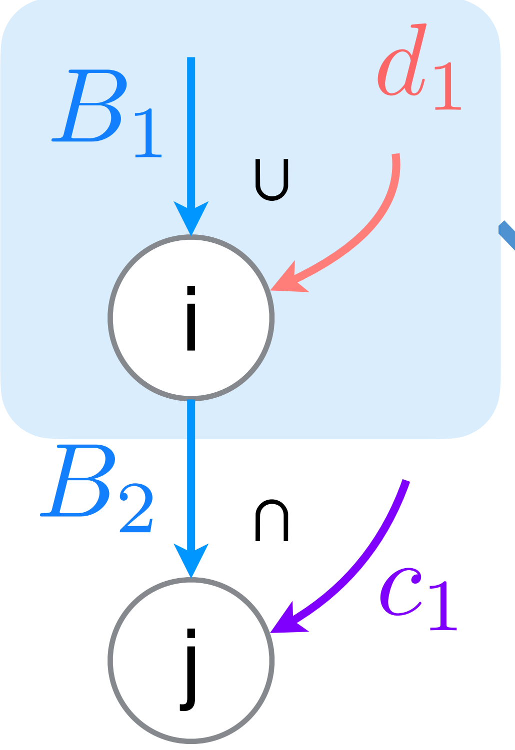




  if (iB == i) iB++;
  if (id == i) id++;
}
```

Recursive lowering of concrete index notation

$$a_i = B_{ij} c_j + d_i$$



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            LOWER($S_{\text{simplified}}$)
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        end for
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        Emit loop footer
    end for
end function
```
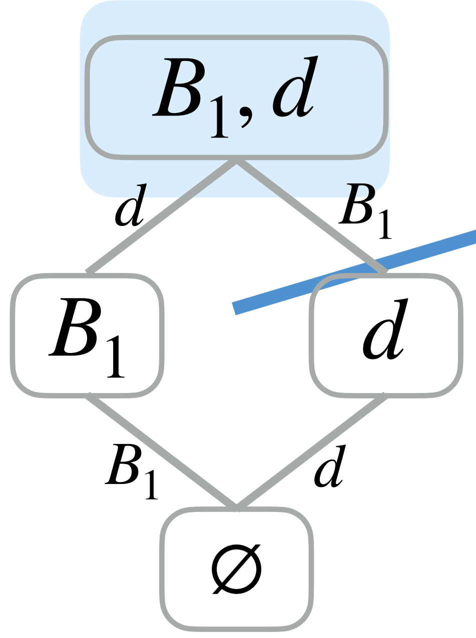
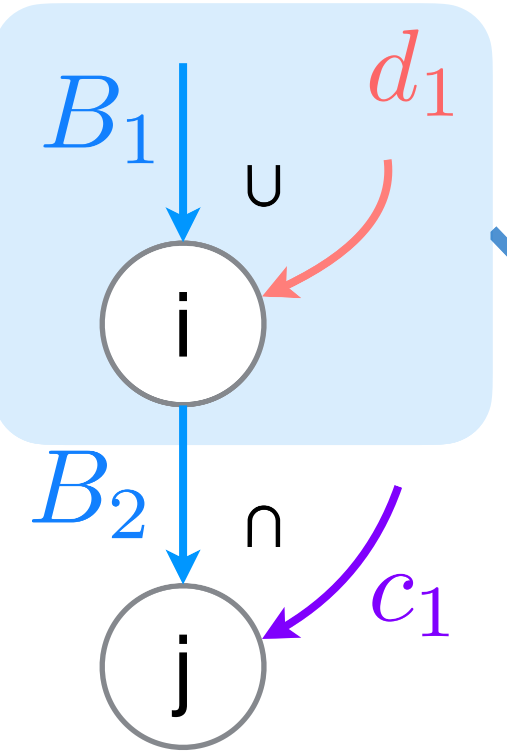



```
int iB = B1_pos[0];
int id = d1_pos[0];
while (iB < B1_pos[1] && id < d1_pos) {
  int iB = B1_crd[iB];
  int id = d1_crd[id];
  int i = min(iB, id);
  if (iB == i && id == i) {



  }
  else if (iB == i) {


  }


  if (iB == i) iB++;
  if (id == i) id++;
}
```

Recursive lowering of concrete index notation

$$a_i = B_{ij} c_j + d_i$$



```
function LOWER(forall statement $S_{\text{forall}}$ of index variable $i$)
    let $\mathcal{L}$ be an iteration lattice constructed from $S_{\text{forall}}$
    Emit initialize iterators
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        end for
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        Emit loop footer
    end for
end function
```
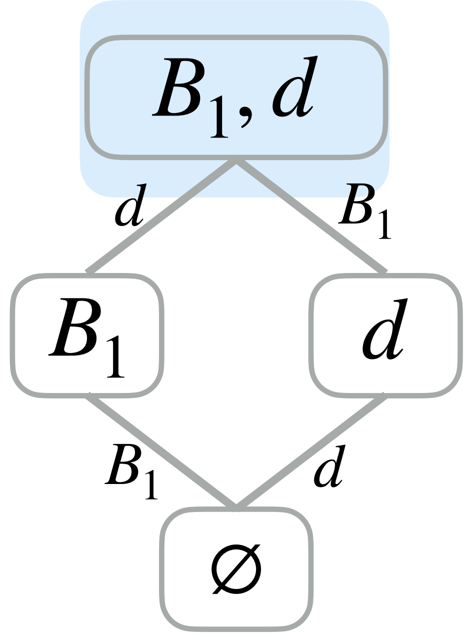
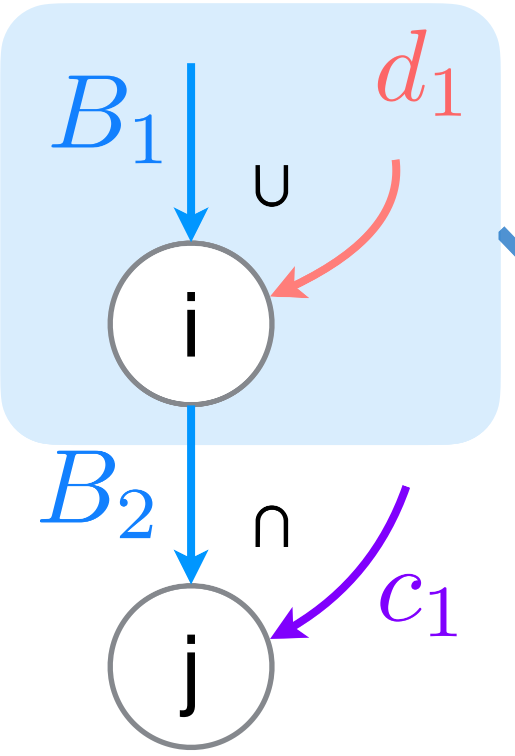
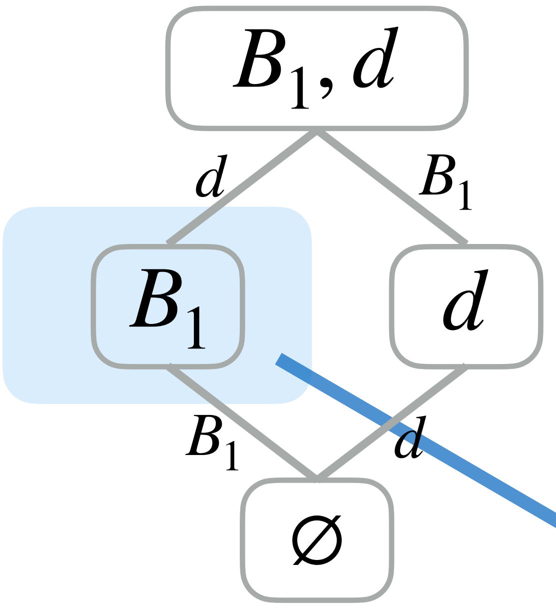



```
int iB = B1_pos[0];
int id = d1_pos[0];
while (iB < B1_pos[1] && id < d1_pos) {
  int iB = B1_crd[iB];
  int id = d1_crd[id];
  int i = min(iB, id);
  if (iB == i && id == i) {







  }
  else if (iB == i) {





  }
  else {
    a[i] = d[id];
  }
  if (iB == i) iB++;
  if (id == i) id++;
}
```

Recursive lowering of concrete index notation

$$a_i = B_{ij} c_j + d_i$$





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        Emit loop header
        Emit access iterators
        Emit resolve the coordinate of $i$
        Emit locate from locators
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            Emit conditional header
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                the body of $S_{\text{forall}}$ by removing operands
                that have run out of values in $\mathcal{L}_q$
            LOWER($S_{\text{simplified}}$)
            Emit conditional footer
        end for
        Emit advance iterators
        Emit loop footer
    end for
end function
```

```
int iB = B1_pos[0];
int id = d1_pos[0];
while (iB < B1_pos[1] && id < d1_pos) {
  int iB = B1_crd[iB];
  int id = d1_crd[id];
  int i = min(iB, id);
  if (iB == i && id == i) {

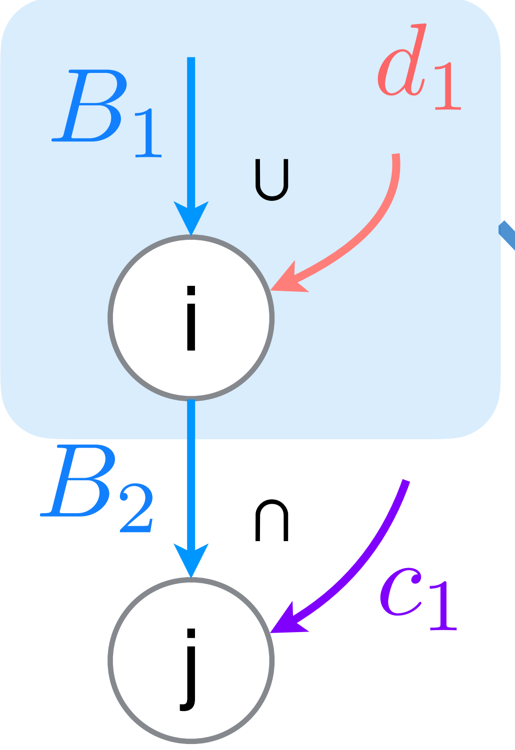
  }
  else if (iB == i) {

  }
  else {
    a[i] = d[id];
  }
  if (iB == i) iB++;
  if (id == i) id++;
}

while (iB < B1_pos[1]) {
  int i = B1_crd[iB];
  for (int jB = B2_pos[iB]; jB < B2_pos[iB+1]; jB++) {
    int j = B2_crd[jB];
    a[i] += B[jB] * c[j];
  }
  iB++;
}
```

Recursive lowering of concrete index notation

$$a_i = B_{ij} c_j + d_i$$



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function LOWER(forall statement $S_{\text{forall}}$ of index variable $i$)
    let $\mathcal{L}$ be an iteration lattice constructed from $S_{\text{forall}}$
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    for each lattice point $\mathcal{L}_p$ in $\mathcal{L}$ do
        Emit loop header
        Emit access iterators
        Emit resolve the coordinate of $i$
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        end for
        Emit advance iterators
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```
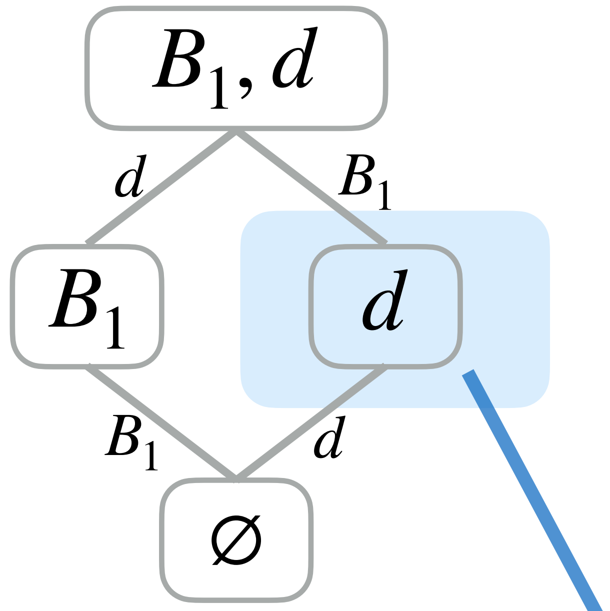
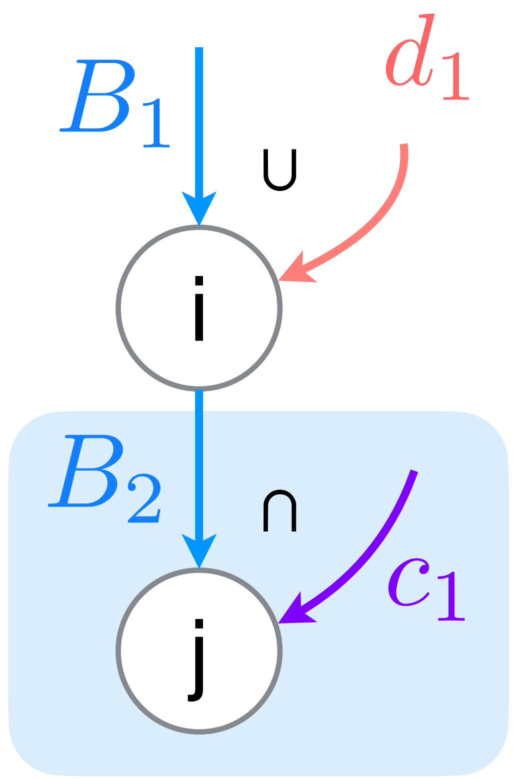



```
int iB = B1_pos[0];
int id = d1_pos[0];
while (iB < B1_pos[1] && id < d1_pos) {
  int iB = B1_crd[iB];
  int id = d1_crd[id];
  int i = min(iB, id);
  if (iB == i && id == i) {






  }
  else if (iB == i) {




  }
  else {
    a[i] = d[id];
  }
  if (iB == i) iB++;
  if (id == i) id++;
}

while (iB < B1_pos[1]) {
  int i = B1_crd[iB];
  for (int jB = B2_pos[iB]; jB < B2_pos[iB+1]; jB++) {
    int j = B2_crd[jB];
    a[i] += B[jB] * c[j];
  }
  iB++;
}

while (id < d1_pos) {
  int i = d1_crd[id];
  a[i] = d[id];
  id++;
}
```

Recursive lowering of concrete index notation

$$a_i = B_{ij}\,c_j + d_i$$



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        Emit loop footer
    end for
end function
```

```
int iB = B1_pos[0];
int id = d1_pos[0];
while (iB < B1_pos[1] && id < d1_pos) {
  int iB = B1_crd[iB];
  int id = d1_crd[id];
  int i = min(iB, id);
  if (iB == i && id == i) {






  }
  else if (iB == i) {





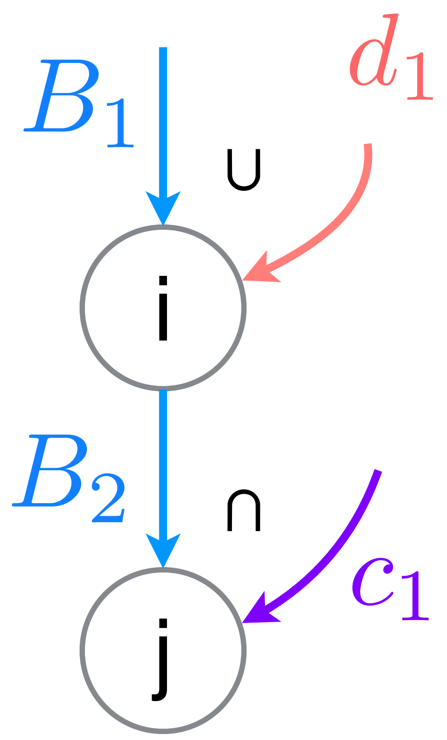
  }
  else {
    a[i] = d[id];
  }
  if (iB == i) iB++;
  if (id == i) id++;
}

while (iB < B1_pos[1]) {
  int i = B1_crd[iB];
  for (int jB = B2_pos[iB]; jB < B2_pos[iB+1]; jB++) {
    int j = B2_crd[jB];
    a[i] += B[jB] * c[j];
  }
  iB++;
}

while (id < d1_pos) {
  int i = d1_crd[id];
  a[i] = d[id];
  id++;
}
```

Recursive lowering of concrete index notation

$$a_i = B_{ij}\, c_j + d_i$$



```
function LOWER(forall statement $S_{\text{forall}}$ of index variable $i$)
    let $\mathcal{L}$ be an iteration lattice constructed from $S_{\text{forall}}$
    Emit initialize iterators
    for each lattice point $\mathcal{L}_p$ in $\mathcal{L}$ do
        Emit loop header
        Emit access iterators
        Emit resolve the coordinate of $i$
        Emit locate from locators
        for each lattice point $\mathcal{L}_q < \mathcal{L}_p$ in $\mathcal{L}$ do
            Emit conditional header
            let $S_{\text{simplified}}$ be a statement constructed from
                the body of $S_{\text{forall}}$ by removing operands
                that have run out of values in $\mathcal{L}_q$
            LOWER($S_{\text{simplified}}$)
            Emit conditional footer
        end for
        Emit advance iterators
        Emit loop footer
    end for
end function
```

$$a_i = B_{ij}\, c_j + d_i$$

```
int iB = B1_pos[0];
int id = d1_pos[0];
while (iB < B1_pos[1] && id < d1_pos) {
  int iB = B1_crd[iB];
  int id = d1_crd[id];
  int i = min(iB, id);
  if (iB == i && id == i) {
    double tj = 0.0;
    for (int jB = B2_pos[iB]; jB < B2_pos[iB+1]; jB++) {
      int j = B2_crd[jB];
      tj += B[jB] * c[j];
    }
    a[i] = tj + d[id];
  }
  else if (iB == i) {



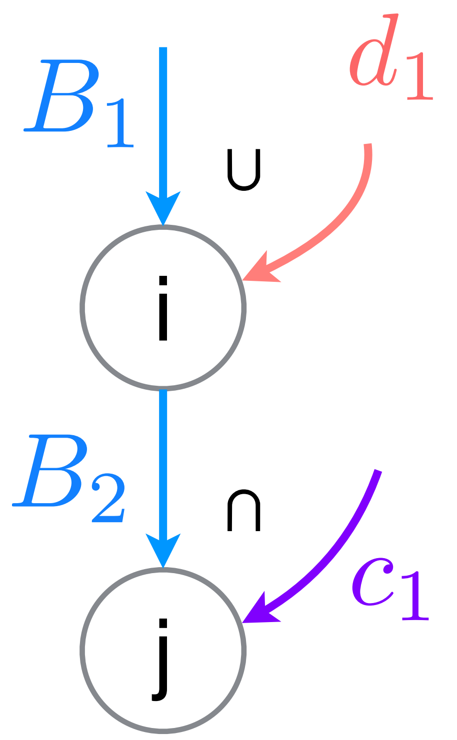

  }
  else {
    a[i] = d[id];
  }
  if (iB == i) iB++;
  if (id == i) id++;
}

while (iB < B1_pos[1]) {
  int i = B1_crd[iB];
  for (int jB = B2_pos[iB]; jB < B2_pos[iB+1]; jB++) {
    int j = B2_crd[jB];
    a[i] += B[jB] * c[j];
  }
  iB++;
}

while (id < d1_pos) {
  int i = d1_crd[id];
  a[i] = d[id];
  id++;
}
```

Recursive lowering of concrete index notation

$$a_i = B_{ij}\,c_j + d_i$$



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    let $\mathcal{L}$ be an iteration lattice constructed from $S_{\text{forall}}$
    Emit initialize iterators
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            LOWER($S_{\text{simplified}}$)
            Emit conditional footer
        end for
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    end for
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```

$$a_i = B_{ij}\,c_j + d_i$$

$$a_i = B_{ij}\,c_j$$

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int id = d1_pos[0];
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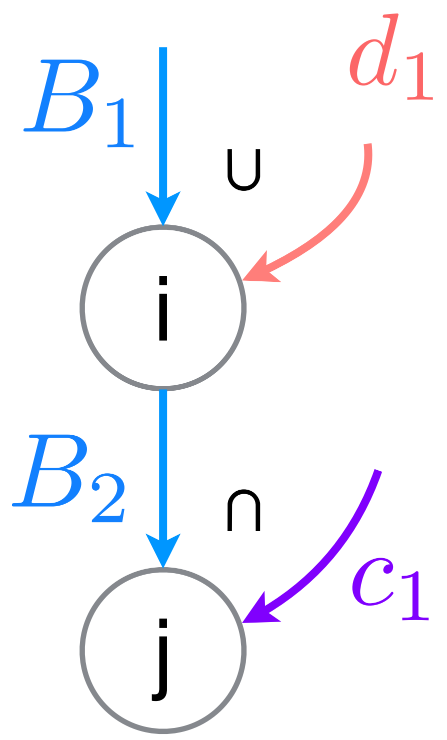
  int id = d1_crd[id];
  int i = min(iB, id);
  if (iB == i && id == i) {
    double tj = 0.0;
    for (int jB = B2_pos[iB]; jB < B2_pos[iB+1]; jB++) {
      int j = B2_crd[jB];
      tj += B[jB] * c[j];
    }
    a[i] = tj + d[id];
  }
  else if (iB == i) {
    for (int jB = B2_pos[iB]; jB < B2_pos[iB+1]; jB++) {
      int j = B2_crd[jB];
      a[i] += B[jB] * c[j];
    }
  }
  else {
    a[i] = d[id];
  }
  if (iB == i) iB++;
  if (id == i) id++;
}

while (iB < B1_pos[1]) {
  int i = B1_crd[iB];
  for (int jB = B2_pos[iB]; jB < B2_pos[iB+1]; jB++) {
    int j = B2_crd[jB];
    a[i] += B[jB] * c[j];
  }
  iB++;
}

while (id < d1_pos) {
  int i = d1_crd[id];
  a[i] = d[id];
  id++;
}
```

Recursive lowering of concrete index notation

$$a_i = B_{ij}\,c_j + d_i$$



```
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            Emit conditional footer
        end for
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    end for
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$$a_i = B_{ij}\,c_j + d_i$$

$$a_i = B_{ij}\,c_j$$

$$a_i = d_i$$

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  int i = min(iB, id);
  if (iB == i && id == i) {
    double tj = 0.0;
    for (int jB = B2_pos[iB]; jB < B2_pos[iB+1]; jB++) {
      int j = B2_crd[jB];
      tj += B[jB] * c[j];
    }
    a[i] = tj + d[id];
  }
  else if (iB == i) {
    for (int jB = B2_pos[iB]; jB < B2_pos[iB+1]; jB++) {
      int j = B2_crd[jB];
      a[i] += B[jB] * c[j];
    }
  }
  else {
    a[i] = d[id];
  }
  if (iB == i) iB++;
  if (id == i) id++;
}

while (iB < B1_pos[1]) {
  int i = B1_crd[iB];
  for (int jB = B2_pos[iB]; jB < B2_pos[iB+1]; jB++) {
    int j = B2_crd[jB];
    a[i] += B[jB] * c[j];
  }
  iB++;
}

while (id < d1_pos) {
  int i = d1_crd[id];
  a[i] = d[id];
  id++;
}
```

Coordinate remapping

```
int ib = b_crd[ib];
int ic = c_crd[ic];
int i = min(ib, ic);
```

```
function LOWER(forall statement $S_{\text{forall}}$ of index variable $i$)
    let $\mathcal{L}$ be an iteration lattice constructed from $S_{\text{forall}}$
    Emit initialize iterators
    for each lattice point $\mathcal{L}_p$ in $\mathcal{L}$ do
        Emit loop header
        Emit access iterators
        Emit map candidate coordinates to the original space
        Emit resolve the coordinate of $i$
        Emit map resolved coordinate to each derived space
        Emit locate from locators
        for each lattice point $\mathcal{L}_q < \mathcal{L}_p$ in $\mathcal{L}$ do
            Emit conditional header
            let $S_{\text{simplified}}$ be a statement constructed from
                the body of $S_{\text{forall}}$ by removing operands
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            LOWER($S_{\text{simplified}}$)
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        end for
        Emit advance iterators
        Emit loop footer
    end for
end function
```

Coordinate remapping

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int ib = b_crd[ib];
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            LOWER($S_{\text{simplified}}$)
            Emit conditional footer
        end for
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end function
```
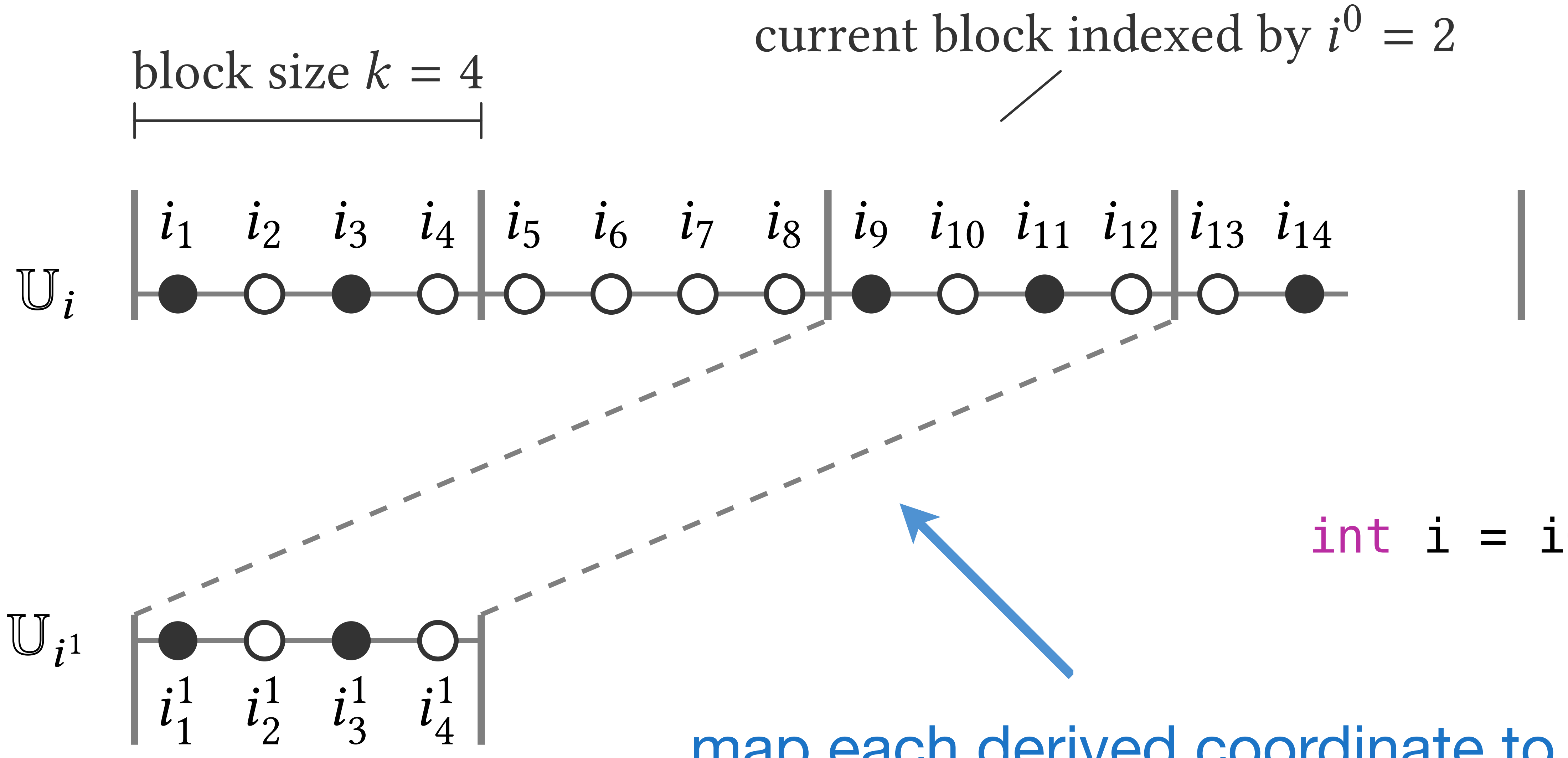

$$\forall_{i_0}\forall_{i_1}\, b_i \cap c_i : i \xrightarrow{\text{split}(\downarrow,4)} i^0 i^1$$



```
int i = i0*4 + i1;
```

map each derived coordinate to original space, so they can be compared